The M-triangle of generalised non-crossing partitions for the types E_7 and E_8

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Abstract. The M-triangle of a ranked locally finite poset P is the generating function $\sum_{u,w\in P}\mu(u,w)\,x^{\operatorname{rk} u}y^{\operatorname{rk} w}$, where $\mu(.,.)$ is the Möbius function of P. We compute the M-triangle of Armstrong's poset of m-divisible non-crossing partitions for the root systems of type E_7 and E_8. For the other types except D_n this had been accomplished in the earlier paper "The F-triangle of the generalised cluster complex" [in: "Topics in Discrete Mathematics," M. Klazar, J. Kratochvil, M. Loebl, J. Matoušek, R. Thomas and P. Valtr, eds., Springer–Verlag, Berlin, New York, 2006, pp. 93–126]. Altogether, this almost settles Armstrong's $F=M$ Conjecture, predicting a surprising relation between the M-triangle of the m-divisible partitions poset and the F-triangle (a certain refined face count) of the generalised cluster complex of Fomin and Reading, the only gap remaining in type D_n. Moreover, we prove a reciprocity result for this M-triangle, again with the possible exception of type D_n. Our results are based on the calculation of certain decomposition numbers for the reflection groups of types E_7 and E_8, which carry in fact finer information than the M-triangle does. The decomposition numbers for the other exceptional reflection groups had been computed in the earlier paper. As an aside, we show that there is a closed form product formula for the type A_n decomposition numbers, leaving the problem of computing the type B_n and type D_n decomposition numbers open.

1. Introduction. The lattice of non-crossing partitions of Kreweras [18] is a now classical object of study in combinatorics with many fascinating properties (see [21]). In [12], Edelman generalised non-crossing partitions to m-divisible non-crossing partitions and showed that they, too, have many beautiful properties. Recently, Bessis [6] and Brady and Watt [8] gave a uniform definition of non-crossing partitions associated to root systems, which includes Kreweras non-crossing partitions as "type A_n" non-crossing partitions,

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as well as Reiner's [20] "type B_n" non-crossing partitions. (As it turned out, Reiner did not have the "right" definition of "type D_n" non-crossing partitions). The question whether Edelman's [12] m-divisible non-crossing partitions also allow for an extension to root systems was answered by Armstrong [2], who introduced m-divisible non-crossing partitions for all root systems uniformly. (He calls them also "generalised" non-crossing partitions for root systems.)

Extending an earlier conjecture of Chapoton [11] (which, in the meantime, has become a theorem due to Athanasiadis [3]), Armstrong predicted a relation between the "M-triangle" of his m-divisible non-crossing partitions poset and the "F-triangle" of the generalised cluster complex of Fomin and Reading [13]. (Roughly speaking, the M-triangle is a bivariate generating function involving the Möbius function, while the F-triangle is a refined face count. The reader is referred to Section 2 for the definition of the M-triangle, and to Section 8 for the definition of the F-triangle.) In the sequel, we shall refer to Armstrong's conjecture as the "$F = M$ Conjecture."

In [17], the F-triangle of the generalised cluster complex was computed for all types, and the M-triangle of the m-divisible non-crossing partitions poset was computed for all types except for the types D_n, E_7, and E_8. Thus, the $F = M$ Conjecture could be verified for all types except the afore-mentioned three. The purpose of this paper is to compute the M-triangle of the m-divisible non-crossing partitions poset for the exceptional types E_7 and E_8. Thus, the $F = M$ Conjecture remains open only for D_n. It should be noted, however, that a conjectural expression for the M-triangle of the m-divisible non-crossing partitions poset appears in [17, Sec. 11, Prop. D]. Moreover, since, in the case of type D_n, Section 11 of [17] contains a proof for $m = 1$ (independent from the one in [3]), the results from [17] together with the results of the present paper provide a case-by-case proof of the $m = 1$ case of the $F = M$ Conjecture, that is, of Chapoton's (ex-)conjecture.

In order to compute the M-triangle of the m-divisible partitions poset for E_7 and E_8, we compute certain decomposition numbers in the corresponding reflection groups (see Section 3 for the definition). These decomposition numbers carry, in fact, finer information than the M-triangle. They are therefore of intrinsic interest. For the other reflection groups of exceptional type, they had been already computed in [17], but not for the types A_n, B_n, and D_n. In Section 10, we point out that, in fact, a result of Goulden and Jackson [15, Theorem 3.2] on the minimal factorization of a long cycle implies a closed form product formula for the type A_n decomposition numbers. We plan to address the problem of computing the type B_n and type D_n decomposition numbers in a future publication.

Our paper is organised as follows. The next section contains the definition and basics of non-crossing partitions for root systems and the m-divisible non-crossing partitions. There, we define also the main object of this paper, the M-triangle of the m-divisible non-crossing partitions poset. Section 3 recalls the formula from [17] which expresses the M-triangle in terms of the afore-mentioned decomposition numbers and the characteristic polynomials of non-crossing partitions posets of lower rank. Section 4 explains our strategy of how to compute these decomposition numbers for the types E_7 and E_8. (This strategy is actually applicable for any specific reflection group, except that the computational difficulties increase significantly with the rank of the reflection group.) The intermediate Section 5 recalls from [17] how these decomposition numbers can be used to compute the

characteristic polynomial of the non-crossing partitions poset corresponding to the reflection group. The programme from Section 4 is then implemented in Section 6 to compute the M-triangle of the m-divisible non-crossing partitions poset of type E_7 and in Section 7 to compute the M-triangle of the m-divisible non-crossing partitions poset of type E_8. The purpose of Section 8 is to briefly explain the $F = M$ Conjecture, and why the results from Sections 6 and 7 together with results from [17] prove it for the types E_7 and E_8. Section 9 presents a curious observation which results from our explicit expressions in this paper and in [17] for the M-triangle of the m-divisible non-crossing partitions poset: a reciprocity relation which sets the M-triangle with parameter m in relation with the M-triangle with parameter $-m$. It would be interesting to find an intrinsic explanation of this phenomenon. We conclude the paper by addressing the type A_n decomposition numbers. Theorem 9 in Section 10 states the afore-mentioned result by Goulden and Jackson in the language of the present paper, and Theorem 10 gives the implied result on arbitrary type A_n decomposition numbers. An Appendix lists the decomposition numbers for the types $A_1, A_2, A_3, A_4, A_5, A_6, A_7, D_4, D_5, D_6, D_7, E_6$, which are needed in our calculations of Sections 6 and 7.

2. Generalised non-crossing partitions. In this section we recall the definition of Armstrong's [2] m-divisible non-crossing partitions poset, and we define its M-triangle.

Let Φ be a finite root system of rank n. (We refer the reader to [16] for all root system terminology.) For an element $\alpha \in \Phi$, let t_α denote the reflection in the central hyperplane perpendicular to α. Let $W = W(\Phi)$ be the group generated by these reflections. By definition, any element w of W can be represented as a product $w = t_1 t_2 \cdots t_\ell$, where the t_i's are reflections. We call the minimal number of reflections which is needed for such a product representation the *absolute length* of w, and we denote it by $\ell_T(w)$. We then define the *absolute order* on W, denoted by $\le_T$, by

$$u \le_T w \quad \text{if and only if} \quad \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w).$$

It can be shown that this is equivalent to the statement that any shortest product representation of u by reflections occurs as an initial segment in some shortest product representation of w by reflections.

We can now define the *non-crossing partition lattice* $NC(\Phi)$. Let c be a *Coxeter element* in W, that is, the product of all reflections corresponding to the simple roots. Then $NC(\Phi)$ is defined to be the restriction of the partial order $\le_T$ to the set of all elements which are less than or equal to c in absolute order. This definition makes sense because, regardless of the chosen Coxeter element c, the resulting poset is always the same up to isomorphism. It can be shown that $NC(\Phi)$ is in fact a lattice (see [9] for a uniform proof), and moreover self-dual (this is obvious from the definition). Clearly, the minimal element in $NC(\Phi)$ is the identity element in W, which we denote by ε, and the maximal element in $NC(\Phi)$ is the chosen Coxeter element c. The term "non-crossing partition lattice" is used because $NC(A_n)$ is isomorphic to the lattice of non-crossing partitions originally introduced by Kreweras [18] (see also [14]), and because also $NC(B_n)$ and $NC(D_n)$ can be realised as lattices of non-crossing partitions (see [4, 20]).

The poset of *m-divisible non-crossing partitions* has as a ground-set the following subset

of $(NC(\Phi))^{m+1}$,

$$NC^m(\Phi) = \{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and}\\ \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\}. \quad (2.1)$$

The order relation is defined by

$$(u_0; u_1, \dots, u_m) \le (w_0; w_1, \dots, w_m) \quad \text{if and only if} \quad u_i \ge_T w_i,\ 1 \le i \le m.$$

We emphasize that, according to this definition, u_0 and w_0 need not be related in any way. The poset $NC^m(\Phi)$ is graded by the rank function

$$\operatorname{rk}\big((w_0; w_1, \dots, w_m)\big) = \ell_T(w_0).$$

Thus, there is a unique maximal element, namely $(c; \varepsilon, \dots, \varepsilon)$, where ε stands for the identity element in W, but, if $m > 1$, there are several different minimal elements. In particular, there is no global minimum in $NC^m(\Phi)$ if $m > 1$ and, hence, $NC^m(\Phi)$ is not a lattice for $m > 1$. (It is, however, a graded join-semilattice, see [2, Theorem 3.4.4].)

The central object in the present paper is the "M-triangle" of $NC^m(\Phi)$, which is the polynomial defined by

$$M_\Phi^m(x, y) = \sum_{u,w \in NC^m(\Phi)} \mu(u, w)\, x^{\operatorname{rk} u} y^{\operatorname{rk} w},$$

where $\mu(u, w)$ is the Möbius function in $NC^m(\Phi)$. It is called "triangle" because the Möbius function $\mu(u, w)$ vanishes unless $u \le w$, and, thus, the only coefficients in the polynomial which may be non-zero are the coefficients of $x^k y^l$ with $0 \le k \le l \le n$.

An equivalent object is the *dual M-triangle*, which is defined by

$$(M_\Phi^m)^*(x, y) = \sum_{u,w \in (NC^m(\Phi))^*} \mu^*(u, w)\, x^{\operatorname{rk}^* w} y^{\operatorname{rk}^* u},$$

where $(NC^m(\Phi))^*$ denotes the poset dual to $NC^m(\Phi)$ (i.e., the poset which arises from $NC^m(\Phi)$ by reversing all order relations), where μ^* denotes the Möbius function in $(NC^m(\Phi))^*$, and where rk^* denotes the rank function in $(NC^m(\Phi))^*$. It is equivalent since, obviously, we have

$$(M_\Phi^m)^*(x, y) = (xy)^n M_\Phi^m(1/x, 1/y). \quad (2.2)$$

The dual M-triangle of $NC^m(\Phi)$ (and, thus, its M-triangle as well) was computed explicitly in [17] for all types, except for D_n (a conjectural expression appears, however, in [17, Sec. 11, Prop. D]), for E_7, and for E_8. In Sections 6 and 7 below, we fill this gap for E_7 and E_8. Thus, it is only the case of D_n which remains open.

3. How to compute the M-triangle of the generalised non-crossing partitions for a specific root system. We follow the strategy outlined in [17, Sec. 12], which, however, has to be complemented by additional ideas. These additional ideas will be described in the next section.

Let us recall from [17] that the dual M-triangle (and, thus, the M-triangle as well) can be expressed in terms of certain *decomposition numbers* and *characteristic polynomials* of non-crossing partitions posets, which we define now. The decomposition number $N_\Phi(T_1, T_2, \dots, T_d)$ is the number of "minimal" products $c_1 c_2 \cdots c_d$ less than or equal to the Coxeter element c in absolute order, "minimal" meaning that all the c_i's are different from ε and that $\ell_T(c_1) + \ell_T(c_2) + \cdots + \ell_T(c_d) = \ell_T(c_1 c_2 \cdots c_d)$, such that the type of c_i as a parabolic Coxeter element is T_i, $i = 1, 2, \dots, d$. (Here, the term "parabolic Coxeter element" means a Coxeter element in some parabolic subgroup. The reader must recall that it follows from [6, Lemma 1.4.3] that any element c_i is indeed a Coxeter element in a parabolic subgroup of $W = W(\Phi)$. By definition, the type of c_i is the type of this parabolic subgroup.) On the other hand, we denote by $\chi^*_{NC(\Psi)}(y)$ the reciprocal polynomial of the characteristic polynomial of $NC(\Psi)$, that is, using self-duality of $NC(\Psi)$,

$$\chi^*_{NC(\Psi)}(y) = \sum_{u \in NC(\Psi)} \mu(\hat{0}_{NC(\Psi)}, u)\, y^{\operatorname{rk}\Psi - \operatorname{rk} u} = \sum_{u \in NC(\Psi)} \mu(u, \hat{1}_{NC(\Psi)})\, y^{\operatorname{rk} u},$$

where $\hat{0}_{NC(\Psi)}$ stands for the minimal element and $\hat{1}_{NC(\Psi)}$ stands for the maximal element in $NC(\Psi)$. (The reader should recall from Section 2 that $\hat{0}_{NC(\Psi)}$ is the identity element in $W(\Psi)$ and that $\hat{1}_{NC(\Psi)}$ is the chosen Coxeter element in $W(\Psi)$.) Using this notation, a combination of Eqs. (12.3) and (12.4) from [17] reads as follows.

Proposition 1. *For any finite root system Φ of rank n, we have*

$$(M^m_\Phi)^*(x,y) = \sum_{d=0}^{n} \sum_{(T_1,\dots,T_d)} x^{\operatorname{rk} T_1 + \cdots + \operatorname{rk} T_d} \cdot N_\Phi(T_1, T_2, \dots, T_d) \cdot \chi^*_{NC(T_1)}(y) \chi^*_{NC(T_2)}(y) \cdots \chi^*_{NC(T_d)}(y) \binom{m}{d}, \tag{3.1}$$

where the inner sum is over all possible d-tuples $(T_1, T_2, \dots, T_d)$ of types (not necessarily irreducible types). In (3.1), and in the sequel, the notation $NC(T)$ means $NC(\Psi)$, where Ψ is a root system of type T, and $\operatorname{rk} T$ denotes the rank of Ψ.

So, what we have to do to apply Formula (3.1) to compute the (dual) M-triangle is, first, to determine all the decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$, and, taking advantage of the multiplicativity

$$\chi^*_{NC(\Psi_1 * \Psi_2)}(y) = \chi^*_{NC(\Psi_1) \times NC(\Psi_2)}(y) = \chi^*_{NC(\Psi_1)}(y) \cdot \chi^*_{NC(\Psi_2)}(y)$$

of the characteristic polynomial (here, $\Psi_1 * \Psi_2$ denotes the direct sum of the root systems Ψ_1 and Ψ_2), second, one needs a list of the characteristic polynomials $\chi^*_{NC(\Psi)}(y)$ for all *irreducible* root systems Ψ. We describe in the following section how we compute the decomposition numbers, and in the subsequent section how to use this knowledge to compute as well the characteristic polynomials that we need.

4. How to compute the decomposition numbers. Our strategy to compute the decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$ for a fixed root system Φ is to find as many linear relations between them as possible, and eventually solve this system of linear equations. Indeed, the decomposition numbers have many relations between themselves, some of which have already been stated and used in [17]. We recall these here in Proposition 2 below. Equation (4.1) says that the order of the types $T_1, T_2, \dots, T_d$ is not relevant. Equation (4.2) reduces the computation to the computation of the decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$ with "full" rank, that is, with $\operatorname{rk} T_1 + \operatorname{rk} T_2 + \cdots + \operatorname{rk} T_d = \operatorname{rk} \Phi$.

In Equation (4.3) we add another family of relations. They involve the decomposition numbers $N_T(T_1', T_2', \dots, T_e')$ of "smaller" root systems than Φ, that is, decomposition numbers with $\operatorname{rk} T < \operatorname{rk} \Phi$. (Similarly to Proposition 1, by abuse of notation, $N_T(T_1', T_2', \dots, T_e')$ stands for the decomposition number $N_\Psi(T_1', T_2', \dots, T_e')$, where Ψ is a root system of type T.) If we had already computed these beforehand, then the relations (4.3) are linear relations between full rank decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$. Proposition 7 allows one to reduce these beforehand computations to irreducible root systems of smaller rank.

A further set of linear relations comes from an identity featuring the zeta polynomials of (ordinary and generalized) non-crossing partitions posets given in Proposition 3. Indeed, as we explain in Remark (2) after the proof of Proposition 3, all the zeta polynomials appearing in (4.7) are explicitly known, so that, by comparing coefficients of $m^i z^j$ on both sides of (4.7), we obtain a set of linear relations between the decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$.

While the number of equations which result from Propositions 2 and 3 exceeds the number of variables (that is, the number of decomposition numbers of full rank) by far, it turns out that they are not sufficient to determine them uniquely. To remedy this somewhat, we add Proposition 4 which provides the values for three special decomposition numbers, and we add Proposition 6 which, as we illustrate in the Remark after the statement of the proposition, allows one to compute all the (full rank) decomposition numbers $N_\Phi(T, A_1)$ with $\operatorname{rk} T = \operatorname{rk} \Phi - 1$.

As we shall see in Sections 6 and 7, the system of linear equations resulting from Propositions 2, 3, 4 and 6 still does not yield a unique solution for the decomposition numbers for $\Phi = E_7$ and $\Phi = E_8$. However, they allow one to come very close, so that, upon adding some arithmetic considerations and, in type E_8, a *Maple* calculation using Stembridge's `coxeter` package [24], one eventually succeeds in finding all the decomposition numbers.

Proposition 2. *Let Φ be a finite root system. Then, for any permutation σ of $\{1, 2, \dots, d\}$, we have*

$$N_\Phi(T_{\sigma(1)}, T_{\sigma(2)}, \dots, T_{\sigma(d)}) = N_\Phi(T_1, T_2, \dots, T_d). \tag{4.1}$$

Furthermore,

$$N_\Phi(T_1, T_2, \dots, T_d) = \sum_T N_\Phi(T_1, T_2, \dots, T_d, T), \tag{4.2}$$

where the sum is over all types T of rank $\operatorname{rk} \Phi - \operatorname{rk} T_1 - \operatorname{rk} T_2 - \cdots - \operatorname{rk} T_d$.

If $\operatorname{rk} T_1 + \operatorname{rk} T_2 + \dots + \operatorname{rk} T_d + \operatorname{rk} T_1' + \operatorname{rk} T_2' + \dots + \operatorname{rk} T_e' = \operatorname{rk} \Phi$*, then*

$$N_\Phi(T_1, T_2, \dots, T_d, T_1', T_2', \dots, T_e') = \sum_T N_T(T_1', T_2', \dots, T_e') N_\Phi(T_1, T_2, \dots, T_d, T), \qquad (4.3)$$

where the sum is over all types T *of rank* $\operatorname{rk} T_1' + \operatorname{rk} T_2' + \dots + \operatorname{rk} T_e'$*.*

Finally, if one of $T_1, T_2, \dots, T_d$ *is not the type of a sub-diagram of the Dynkin diagram of* Φ*, then* $N_\Phi(T_1, T_2, \dots, T_d) = 0$*.*

Examples. An example of a relation of the form (4.2) is (7.11). To be precise, Equation (7.11) is the special case of (4.2) where $\Phi = E_8$, $d = 1$, and $T_1 = D_4$.

An example of a relation of the form (4.3) is (we make also use of (4.1))

$$\begin{aligned} N_{E_7}(A_1 * A_3, A_1^2, A_1) = {} & N_{A_1^5}(A_1 * A_3, A_1) N_{E_7}(A_1^5, A_1^2) \\ & + N_{A_1^3 * A_2}(A_1 * A_3, A_1) N_{E_7}(A_1^3 * A_2, A_1^2) \\ & + N_{A_1 * A_2^2}(A_1 * A_3, A_1) N_{E_7}(A_1 * A_2^2, A_1^2) \\ & + N_{A_1^2 * A_3}(A_1 * A_3, A_1) N_{E_7}(A_1^2 * A_3, A_1^2) \\ & + N_{A_2 * A_3}(A_1 * A_3, A_1) N_{E_7}(A_2 * A_3, A_1^2) \\ & + N_{A_1 * A_4}(A_1 * A_3, A_1) N_{E_7}(A_1 * A_4, A_1^2) \\ & + N_{A_1 * D_4}(A_1 * A_3, A_1) N_{E_7}(A_1 * D_4, A_1^2) \\ & + N_{A_5}(A_1 * A_3, A_1) N_{E_7}(A_5, A_1^2) \\ & + N_{D_5}(A_1 * A_3, A_1) N_{E_7}(D_5, A_1^2) \\ = {} & 2N_{E_7}(A_1^2 * A_3, A_1^2) + 3N_{E_7}(A_2 * A_3, A_1^2) + 5N_{E_7}(A_1 * A_4, A_1^2) \\ & + 9N_{E_7}(A_1 * D_4, A_1^2) + 6N_{E_7}(A_5, A_1^2) + 4N_{E_7}(D_5, A_1^2). \end{aligned}$$

To be precise, the above equation is the special case of (4.3) where $\Phi = E_7$, $d = 1$, $e = 2$, $T_1 = A_1^2$, $T_1' = A_1 * A_3$ and $T_2' = A_1$. The decomposition numbers $N_\Psi(\dots)$ with $\Psi = A_1^5, A_1^3 * A_2, A_1 * A_2^2, A_1^2 * A_3, A_2 * A_3, A_1 * A_4, A_1 * D_4, A_5, D_5$ which are needed here are computed by using Proposition 7 and the decomposition numbers for the irreducible root systems $A_1, A_2, A_3, A_4, A_5, D_4, D_5$ tabulated in the Appendix.

Proof of Proposition 2. Equation (4.1) is [17, Eq. (45)], while Equation (4.2) is [17, Eq. (46)].

In order to see (4.3), we recall that, by definition and by taking into account the rank assumption, the number $N_\Phi(T_1, T_2, \dots, T_d, T_1', T_2', \dots, T_e')$ is the number of decompositions

$$c = c_1 c_2 \cdots c_d c_1' c_2' \cdots c_e', \qquad (4.4)$$

with c_i of type T_i and c_i' of type T_i' for all i. The decompositions (4.4) can also be determined by first decomposing

$$c = c_1 c_2 \cdots c_d c', \qquad (4.5)$$

with c_i of type T_i for all i, and subsequently

$$c' = c'_1 c'_2 \cdots c'_e, \tag{4.6}$$

with c'_i of type T'_i for all i. If we fix the type of c', T say, then the number of decompositions (4.5) is $N_\Phi(T_1, T_2, \dots, T_d, T)$. For determining the number of decompositions (4.6), we recall from [6, Cor. 1.6.2 and Def. 1.6.3] that c' is a Coxeter element in a parabolic subgroup, denoted by $W_{c'}$ in [6], and that the absolute order $\le_T$ on $W = W(\Phi)$ restricted to $W_{c'}$ is identical with absolute order on $W_{c'}$. In other words, the decompositions (4.6) with $c'_i \in W$ are identical with the decompositions (4.6) with $c'_i \in W_{c'}$. Hence, the number of decompositions (4.6) is the number $N_T(T'_1, T'_2, \dots, T'_e)$. Finally, in order to get the *total* number $N_\Phi(T_1, T_2, \dots, T_d)$, we must sum the products

$$N_T(T'_1, T'_2, \dots, T'_e) N_\Phi(T_1, T_2, \dots, T_d, T)$$

over all possible types T, thus arriving at Equation (4.3).

The last assertion follows again from the fact [6, Lemma 1.4.3] that any element which is less than or equal to a Coxeter element c of W is the Coxeter element in some parabolic subgroup of W. Hence, its type must by a sub-type of Φ, the type of W. $\square$

More equations come from the following proposition, featuring the zeta polynomial of the non-crossing partitions posets (ordinary and generalized). Recall that, given a poset P, its *zeta polynomial* $Z_P(z)$ is the number of multichains $x_1 \le x_2 \le \cdots \le x_{z-1}$ in P. (It can be shown that this is indeed a polynomial in z. The reader should consult [23, Sec. 3.11] for more information on this topic.)

Proposition 3. *For any finite root system Φ of rank n, we have*

$$Z_{NC^m(\Phi)}(z) = \sum_{d=0}^{n} \sum_{(T_1,\dots,T_d)} N_\Phi(T_1, T_2, \dots, T_d) \\ \cdot Z_{NC(T_1)}(z-1) Z_{NC(T_2)}(z-1) \cdots Z_{NC(T_d)}(z-1) \binom{m}{d}, \tag{4.7}$$

where the inner sum is over all possible d-tuples $(T_1, T_2, \dots, T_d)$ of types, and where $N_\Phi(T_1, T_2, \dots, T_d)$ is as before.

Proof. Let $\hat{1}_{NC^m(\Phi)}$ denote the maximal element $(c; \varepsilon, \dots, \varepsilon)$ in $NC^m(\Phi)$. If, given a multichain $x_1 \le x_2 \le \cdots \le x_{z-1}$ in $NC^m(\Phi)$, we remove the minimal element $w = x_1$, then a multichain remains which is by 1 shorter. Hence, by summing over all possible w's, we obtain

$$Z_{NC^m(\Phi)}(z) = \sum_{w \in NC^m(\Phi)} Z_{[w, \hat{1}_{NC^m(\Phi)}]}(z-1). \tag{4.8}$$

Now, by definition of $NC^m(\Phi)$, w is of the form $(w_0; w_1, \dots, w_m)$ with $w_0 w_1 \cdots w_m = c$ and $\ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c) = n$. Moreover, as we already recalled in Section 3, it follows from [6, Lemma 1.4.3] that any element w_j is a Coxeter element in a

parabolic subgroup of $W = W(\Phi)$ (or, in the language used earlier, a "parabolic Coxeter element"). On the other hand, we have

$$[w, \hat{1}_{NC^m(\Phi)}] \cong [\varepsilon, w_1] \times [\varepsilon, w_2] \times \cdots \times [\varepsilon, w_m], \tag{4.9}$$

where each interval $[\varepsilon, w_j]$ is an interval in $NC(\Phi)$. In fact, some of the w_j's may be equal to the identity ε. If we denote by $w_{i_1}, w_{i_2}, \dots, w_{i_d}$ those among the w_j's which are *not* equal to ε, then (4.9) reduces to

$$[w, \hat{1}_{NC^m(\Phi)}] \cong [\varepsilon, w_{i_1}] \times [\varepsilon, w_{i_2}] \times \cdots \times [\varepsilon, w_{i_d}].$$

More precisely, since each w_{i_j} is a parabolic Coxeter element, each interval $[\varepsilon, w_{i_j}]$ is isomorphic to some non-crossing partition lattice $NC(\Psi)$, where Ψ is the root system of this parabolic subgroup. The zeta polynomial being multiplicative, this implies

$$Z_{[w,\hat{1}_{NC^m(\Phi)}]}(z-1) = Z_{[\varepsilon,w_{i_1}]}(z-1) Z_{[\varepsilon,w_{i_2}]}(z-1) \cdots Z_{[\varepsilon,w_{i_d}]}(z-1).$$

If we take into account that the number of possibilities to choose the indices $\{i_1 < i_2 < \cdots < i_d\}$ out of $\{1, 2, \dots, m\}$ is equal to $\binom{m}{d}$ and combine this with the previous observations, then (4.8) turns into (4.7). $\square$

Remarks. (1) It is striking to note the similarities between (3.1) and (4.7). It would be interesting to find an intrinsic explanation. Even in lack of such an explanation, the relations (3.1) and (4.7) underline the significance of the decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$.

(2) The zeta polynomial of the non-crossing partition lattice $NC(\Phi)$, where Φ is a root system of rank n, has the elegant formula (see [11, Prop. 9])

$$Z_{NC(\Phi)}(z) = \prod_{i=1}^{n} \frac{(z-1)h + d_i}{d_i}, \tag{4.10}$$

where h is the *Coxeter number* and the d_i's are the *degrees* of $W = W(\Phi)$. (The reader should be warned that the convention chosen in [11] for the zeta polynomial is different from the one here.) This uniform formula was originally conjectured by Chapoton on the basis of the already known formulae in types A_n, B_n and D_n, and of calculations he did for some exceptional groups. The formula was finally confirmed for the large exceptional groups by *Mathematica* and *MATLAB* calculations done by Reiner.

On the basis of (4.10), Armstrong could also determine the zeta polynomial for all m-divisible non-crossing partitions posets. Again, there is a uniform formula, namely (see [2, Theorem 3.5.2]; we warn the reader that also in [2] this different convention for the zeta polynomial is used)

$$Z_{NC^m(\Phi)}(z) = \prod_{i=1} \frac{(z-1)mh + d_i}{d_i}. \tag{4.11}$$

If we use formulae (4.10) and (4.11) in (4.7), then, by comparing coefficients of $m^i z^j$ on both sides of (4.7), we obtain $(n+1)^2$ linear equations for the decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$. Although some of them turn out to be trivial ($1 = 1$ or $0 = 0$), this is nevertheless a considerable number of linear relations.

Proposition 4. (1) *For any finite root system Φ of type T we have $N_\Phi(T) = 1$.*

(2) *The decomposition number $N_\Phi(A_1)$ is equal to the total number of reflections in $W(\Phi)$ or, equivalently, to the number of positive roots in Φ. These numbers are tabulated in* [16, Table 2 on p. 80].

(3) *The decomposition number $N_\Phi(A_1, A_1, \dots, A_1)$ (with $n = \operatorname{rk}\Phi$ occurrences of A_1) is equal to $n!\,h^n/|W(\Phi)|$, where h is the Coxeter number of $W(\Phi)$.*

Proof. The claim (1) is trivial.

To see (2), we have to show that any reflection t is less than or equal to the Coxeter element c in absolute order. Indeed, we have $c = t(tc)$, with $\ell_T(t) = 1$ and $\ell_T(tc) = \ell_T(c) \pm 1$, by general properties of the absolute length. Now, the maximal possible absolute length of an element in $W(\Phi)$ is $\operatorname{rk}\Phi = \ell_T(c)$ (cf. [6, Lemma 1.2.1(ii)]). Hence, we have $\ell_T(tc) = \ell_T(c) - 1$, and, thus, $t \le_T c$.

Finally we prove claim (3). The number of decompositions $c = t_1 t_2 \cdots t_n$ is equal to the number of maximal chains $\varepsilon < w_1 < w_2 < \cdots < w_{n-1} < c$ in $NC(\Phi)$, because of the obvious bijection between decompositions and maximal chains defined by the identification $w_i = t_1 t_2 \dots t_i$, $i = 1, 2, \dots, n-1$. It is a general fact (see [23, Prop. 3.11.1]) that the number of maximal chains in a poset is equal to the leading coefficient of its zeta polynomial multiplied by the factorial of the rank of the poset. The claim then follows from the explicit formula (4.10) for the zeta polynomial of $NC(\Phi)$. $\square$

Our next goal is to determine the decomposition numbers $N_\Phi(T, A_1)$ with $\operatorname{rk} T = \operatorname{rk}\Phi - 1$. Proposition 6 will provide the means to do that. For the proof of the proposition, we need an auxiliary lemma, Lemma 5 below. It is an extended version of Lemma 1.3.4 from [6]. The extension makes the theoretical assertion of [6, Lemma 1.3.4] concrete for each type. Actually, in the present paper, we shall need this "concretisation" only for the types D_6 and E_7. However, as it may be useful in other situations, we work it out here for all types.

Lemma 5. *Let Φ be an irreducible root systems of rank n. Furthermore, let*

$$S = \{s_1, s_2, \dots, s_n\} = \{s_1, \dots, s_r\} \cup \{s_{r+1}, \dots, s_n\}$$

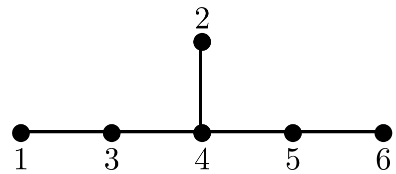
be a choice of simple reflections with the property that s_i and s_j commute for all i, j with $1 \le i, j \le r$, and for all i, j with $r + 1 \le i, j \le n$, r being chosen appropriately (cf. [16, Sec. 3.17]*), and let $c = s_1 s_2 \cdots s_n$ be the corresponding Coxeter element. Then, for any reflection t, the cardinality of the orbit $\Omega(t) = \{c^k t c^{-k} : k \in \mathbb{Z}\}$ of t under conjugation by c is*

(1) *h if $|\Omega(t) \cap S| = 2$,*
(2) *$h/2$ if $|\Omega(t) \cap S| = 1$,*

where h is the Coxeter number of $W = W(\Phi)$ (cf. [16, Table 2 on p. 80]*), and there are no other possibilities. Specifically, the cardinality of $\Omega(t)$ is*

$$\begin{cases} h & \text{if } \Phi = A_n \text{ except if } n \text{ is odd and } tc \text{ is of type } A^2_{(n-1)/2}, \\ & \text{if } \Phi = D_n,\ n \text{ is odd, and } tc \text{ is of type } A_{n-1}, \\ & \text{if } \Phi = E_6 \text{ and } tc \text{ is of type } D_5 \text{ or } A_1 * A_4, \\ h/2 & \text{otherwise.} \end{cases}$$

Proof. The assertion on the cardinality of $\Omega(t)$ as it depends on the cardinality of the intersection $\Omega(t) \cap S$ is [6, Lemma 1.3.4]. The concrete cardinality assertion for the type A_n can be worked out by using the well-known combinatorial realisation of $W(A_n)$ as the symmetric group on $n+1$ elements, while for the types B_n and D_n this can be worked out by using the combinatorial realisations of the corresponding reflection groups as subgroups of the symmetric group on $2n$ elements (see e.g. [7, Sections 8.1 and 8.2].) Since this does not contain any surprises, we leave the details to the reader.



Dynkin diagram of E_6

Figure 1

For the type E_6 we argue as follows. Let $s_1, s_2, s_3, s_4, s_5, s_6$ denote the simple reflections of E_6, with commutation relations coded by the Dynkin diagram of E_6 as given by Figure 1. (The labelling of the nodes is the one which Stembridge's `coxeter` package [24] chooses.) To prepare the following arguments, we note that if T is the type of the sub-diagram of the Dynkin diagram of E_6 obtained by deleting the node corresponding to the simple reflection s_i, say, then, because of $c = s_1s_2s_3s_4s_5s_6 = t(s_1 \cdots s_{i-1}s_{i+1} \cdots s_6)$ with $t = (s_1 \cdots s_{i-1})s_i(s_1 \cdots s_{i-1})^{-1}$, there is (at least) one orbit $\{c^k tc^{-k} : k \in \mathbb{Z}\}$ such that the type of tc is T. The types which occur as types of sub-diagrams of the Dynkin diagram of E_6 obtained by deleting one node from it are D_5, $A_1 * A_4$, A_5 and $A_1 * A_2^2$.

Maple computations using Stembridge's package yield the following facts. There are four orbits of reflections under conjugation by c, two of size $12 = h$, and two of size $6 = h/2$. Hence, for each of the types D_5, $A_1 * A_4$, A_5, and $A_1 * A_2^2$, there corresponds exactly one orbit in the sense described above, that is, if T is one of these four types then there is exactly one orbit such that tc is of type T for each reflection t in the orbit.

The orbit of s_1 has 12 elements, and it contains also s_3, s_4, s_5, s_6. From the fact that it contains s_1, we can conclude that tc is of type D_5 for all reflections t in this orbit. The second orbit with 12 elements is the one of the reflection $s_2s_3s_4s_2s_3$. Since the order of $s_2s_3s_4s_2s_3c$ is 10, we must necessarily have that its type is $A_1 * A_4$. (The other remaining sub-diagrams of the Dynkin diagram of E_6 have types A_5 and $A_1 * A_2^2$. Coxeter elements of these types have the order 6.) The orbits with 6 elements are the ones of s_2 and $s_4s_5s_6s_5s_4$, respectively. If t denotes any reflection in one of these two orbits, then the type of tc is necessarily either A_5 or $A_1 * A_2^2$, one type applying for *all* reflections in one orbit, the other type applying for *all* reflections in the other orbit. For our purpose it is not important which type is associated to which orbit, since both possibilities are in agreement with our claim.

In the case that $\Phi = I_2(a)$, there is just one orbit. For all other exceptional types, one can again do calculations using Stembridge's `coxeter` package. The result is that in type H_3 one obtains 3 orbits of 5 elements each, in type H_4 one obtains 4 orbits of 15 elements each, in type F_4 one obtains 4 orbits of 6 elements each, in type E_7 one obtains 7 orbits

of 9 elements each, and in type E_8 one obtains 8 orbits of 15 elements each. All this is in accordance with our claim. $\square$

Proposition 6. *Let Φ be a finite irreducible root system, and let T be a type with $\operatorname{rk} T = \operatorname{rk} \Phi - 1$. Then, if T is the type of a sub-diagram of the Dynkin diagram of Φ obtained by deleting one node from it, we have*

$$\frac{N_\Phi(T, A_1)}{N(T \subseteq \Phi)} = \frac{h}{2}, \tag{4.12}$$

where $N(T \subseteq \Phi)$ denotes the number of times T arises as type of a sub-diagram of the Dynkin diagram of Φ, and where h is the Coxeter number of $W = W(\Phi)$. Otherwise, we have $N_\Phi(T, A_1) = 0$.

Proof. We start with the following observation. By definition, the number $N_\Phi(T, A_1)$ counts the number of product decompositions $c = wt$, where w is a parabolic Coxeter element of type T and t is a reflection. Given w and t, we obtain another product decomposition by conjugation,

$$c = ccc^{-1} = cwtc^{-1} = (cwc^{-1})(ctc^{-1}),$$

where ctc^{-1} is also a reflection, and where the type of cwc^{-1} is still T. Hence, all the elements of the complete orbit $\{(c^k wc^{-k}, c^k tc^{-k}) : k \in \mathbb{Z}\}$ of (w, t) under conjugation by c provide product decompositions of c in a parabolic Coxeter element of type T and a reflection. Since the reflection t' already determines its "companion" w' in the decomposition $c = w't'$ uniquely via $w' = ct'$, we may as well concentrate on the orbit $\{c^k tc^{-k} : k \in \mathbb{Z}\}$ of t under conjugation by c.

In what follows, we assume the setup of Lemma 5, that is, we assume that

$$S = \{s_1, s_2, \dots, s_n\} = \{s_1, \dots, s_r\} \cup \{s_{r+1}, \dots, s_n\}$$

is a choice of simple reflections with the property that s_i and s_j commute for all i, j with $1 \le i, j \le r$, and for all i, j with $r + 1 \le i, j \le n$, r being chosen appropriately, and we let $c = s_1 s_2 \cdots s_n$ be the corresponding Coxeter element.

Now, let T be the type of the sub-diagram of the Dynkin diagram of Φ obtained by deleting the node corresponding to s_i from it. If $1 \le i \le r$, we have

$$\begin{aligned} c &= \big(s_1 \cdots s_{i-1} s_{i+1} \cdots s_r (s_i s_{r+1} s_i) \cdots (s_i s_n s_i)\big) \cdot s_i \\ &= \big((s_i s_1 s_i) \cdots (s_i s_{i-1} s_i)(s_i s_{i+1} s_i) \cdots (s_i s_n s_i)\big) \cdot s_i, \end{aligned} \tag{4.13}$$

and, if $r + 1 \le i \le n$, we have

$$c = \big(s_1 \cdots s_{i-1} s_{i+1} \cdots s_n\big) \cdot s_i.$$

In both cases, the type of cs_i is T, in the former case since the system

$$\{(s_i s_1 s_i), \dots, (s_i s_{i-1} s_i)(s_i s_{i+1} s_i), \dots, (s_i s_n s_i)\} \tag{4.14}$$

is conjugate to $\{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n\}$, the latter being a system of simple reflections of type T.

Next, let $\Omega_1, \Omega_2, \dots, \Omega_a$ and $\bar\Omega_1, \bar\Omega_2, \dots, \bar\Omega_b$ be the orbits of reflections t under conjugation by c such that ct is of type T,[1] the former those of cardinality h, the latter those of cardinality $h/2$. Lemma 5 says that there can be no others, while the argument involving (4.13) and (4.14) implies that there is at least one such orbit, namely the orbit of s_i. The last statement implies in particular that at least one of a and b is non-zero. We have

$$N(T, A_1) = ah + b\frac{h}{2}. \tag{4.15}$$

On the other hand, from Lemma 5 (1),(2) and the argument involving (4.13) and (4.14) we infer that

$$N(T \subseteq \Phi) = 2a + b. \tag{4.16}$$

Dividing right-hand and left-hand sides, respectively, of (4.15) and (4.16) we obtain (4.12).

If T is a type which is *not* the type of a sub-diagram of the Dynkin diagram of Φ and $N(T, A_1) \neq 0$, then we obtain a contradiction: let t be a reflection such that $c = wt$, where w is of type T. By Lemma 5, the orbit $\Omega(t)$ of t under conjugation by c contains a simple reflection, s_i say. Now the argument involving (4.13) and (4.14) yields that T is the type of a sub-diagram of the Dynkin diagram of Φ, which is absurd. $\square$

The final proposition in this section reduces the computation of the decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$ to irreducible root systems Φ. On the right-hand side of (4.18) below, we use an extended definition of decomposition numbers, where we allow some of the types T_i in the argument to be empty, which we denote by $T_i = \emptyset$: we set $N_\Phi(\emptyset, T_2, \dots, T_d) = N_\Phi(T_2, \dots, T_d)$, with analogous conventions if one or more of the other T_i's should be empty.

Proposition 7. *If*

$$\operatorname{rk}\Phi_1 + \operatorname{rk}\Phi_2 = \operatorname{rk}T_1 + \operatorname{rk}T_2 + \cdots + \operatorname{rk}T_d, \tag{4.17}$$

then

$$N_{\Phi_1 * \Phi_2}(T_1, T_2, \dots, T_d) = \sum_{T_1' * T_1'' = T_1, \dots, T_d' * T_d'' = T_d} N_{\Phi_1}(T_1', T_2', \dots, T_d') \cdot N_{\Phi_2}(T_1'', T_2'', \dots, T_d''), \tag{4.18}$$

where in the sum on the right-hand side any of $T_1', T_2', \dots, T_d', T_1'', T_2'', \dots, T_d''$ could also be empty.

Remark. The reader should note that, in order to have a non-vanishing summand in the sum on the right-hand side of (4.18), we must necessarily have $\operatorname{rk}\Phi_1 = \operatorname{rk}T_1' + \operatorname{rk}T_2' + \cdots + \operatorname{rk}T_d'$ and $\operatorname{rk}\Phi_2 = \operatorname{rk}T_1'' + \operatorname{rk}T_2'' + \cdots + \operatorname{rk}T_d''$, because otherwise one of $N_{\Phi_1}(T_1', T_2', \dots, T_d')$ or $N_{\Phi_2}(T_1'', T_2'', \dots, T_d'')$ vanishes.

[1] A case-by-case analysis shows that there can be at most 2 such orbits, and 2 orbits only if Φ is of type D_n, n is even, and $T = A_{n-1}$. However, this is of no importance here.

Proof of Proposition 7. By definition of the decomposition numbers and by (4.17), the number $N_{\Phi_1*\Phi_2}(T_1,T_2,\dots,T_d)$ is equal to the number of products

$$c_1c_2\cdots c_d=c, \tag{4.19}$$

where c denotes the fixed Coxeter element of type $\Phi_1*\Phi_2$, such that $\ell_T(c_1)+\ell_T(c_2)+\dots+\ell_T(c_d)=\ell_T(c)$, and such that the type of c_i as a parabolic Coxeter element is T_i, $i=1,2,\dots,d$. Each of $c,c_1,c_2,\dots,c_d$ decomposes uniquely into a product of an element of $W(\Phi_1)$ and an element of $W(\Phi_2)$, say $c=c'c''$, $c_1=c_1'c_1''$, $c_2=c_2'c_2''$, $\dots$, $c_d=c_d'c_d''$ where $c',c_1',c_2',\dots,c_d'\in W(\Phi_1)$ and $c'',c_1'',c_2'',\dots,c_d''\in W(\Phi_2)$. Since elements of $W(\Phi_1)$ commute with elements of $W(\Phi_2)$ in $W(\Phi_1*\Phi_2)=W(\Phi_1)\times W(\Phi_2)$, the relation (4.19) is equivalent to the two relations

$$c_1'c_2'\cdots c_d'=c', \tag{4.20a}$$
$$c_1''c_2''\cdots c_d''=c''. \tag{4.20b}$$

Hence, we may compute $N_{\Phi_1*\Phi_2}(T_1,T_2,\dots,T_d)$ also by first determining the number of all possible products (4.20a) such that the type of c_i' is T_i' (where we cover the case that c_i' is the identity element in $W(\Phi_1)$ by declaring that in that case $T_i'=\emptyset$) and the number of all possible products (4.20b) such that the type of c_i'' is T_i'' (with the same convention concerning identity elements), taking their product, and summing the results over all possible $T_1',\dots,T_d',T_1'',\dots,T_d''$ with $T_1'*T_1''=T_1$, $\dots$, $T_d'*T_d''=T_d$. This leads exactly to the right-hand side of (4.18). $\square$

5. How to compute the characteristic polynomials. Aside from the decomposition numbers, the second ingredient which we need for applying (3.1) to compute the (dual) M-triangle of the m-divisible partitions posets $NC^m(\Phi)$ is a list of the characteristic polynomials $\chi^*_{NC(\Psi)}(y)$ for all *irreducible* root systems Ψ of rank at most the rank of Φ. (By the multiplicativity of the characteristic polynomial, this then gives also formulae for the characteristic polynomials of all the reducible types.) In fact, the numbers $N_\Psi(T_1,T_2,\dots,T_d)$ carry all the information which is necessary to do this recursively. Namely, by the definition of $NC(\Psi)$ and of the decomposition numbers $N_\Psi(T_1,T_2,\dots,T_d)$, we have

$$\chi^*_{NC(\Psi)}(y)=\sum_{T_1,T_2}N_\Psi(T_1,T_2)\,\mu_{NC(T_2)}\big(\hat0_{NC(T_2)},\hat1_{NC(T_2)}\big)\,y^{\operatorname{rk}T_1}, \tag{5.1}$$

where $\mu_{NC(T_2)}(.,.)$ denotes the Möbius function in $NC(T_2)$, and where $\hat0_{NC(T_2)}$ and $\hat1_{NC(T_2)}$ are, respectively, the minimal and the maximal element in $NC(T_2)$. Indeed, inductively, the Möbius functions $\mu_{NC(T_2)}\big(\hat0_{NC(T_2)},\hat1_{NC(T_2)}\big)$ are already known for all T_2 of lower rank than the rank of Ψ. Hence, the only unknown in (5.1) is $\mu_{NC(\Psi)}\big(\hat0_{NC(\Psi)},\hat1_{NC(\Psi)}\big)$. However, the latter can be computed by setting $y=1$ in (5.1) and using the fact that $\chi^*_{NC(\Psi)}(1)=0$ for all root systems Ψ of rank at least 1. (This fact is equivalent to the statement that $\sum_{u\in NC(\Psi)}\mu_{NC(\Psi)}\big(u,\hat1_{NC(\Psi)}\big)=0$, which is nothing but a part of the definition of the Möbius function. Alternatively, one may use the uniform formula (4.10)

for the zeta polynomial of the non-crossing partition lattices, in which one specializes the variable to -1, cf. [23, Sec. 3.11].)

Below we list the values of the characteristic polynomials of the irreducible root systems that we need in Sections 6 and 7 (and also for the computations which are behind the numbers in the Appendix).

$$
\begin{aligned}
\chi^*_{A_1}(y) &= y-1,\\
\chi^*_{A_2}(y) &= y^2-3y+2,\\
\chi^*_{A_3}(y) &= y^3-6y^2+10y-5,\\
\chi^*_{A_4}(y) &= y^4-10y^3+30y^2-35y+14,\\
\chi^*_{A_5}(y) &= y^5-15y^4+70y^3-140y^2+126y-42,\\
\chi^*_{A_6}(y) &= y^6-21y^5+140y^4-420y^3+630y^2-462y+132,\\
\chi^*_{A_7}(y) &= y^7-28y^6+252y^5-1050y^4+2310y^3-2772y^2+1716y-429,\\
\chi^*_{D_4}(y) &= y^4-12y^3+39y^2-48y+20,\\
\chi^*_{D_5}(y) &= y^5-20y^4+106y^3-230y^2+220y-77,\\
\chi^*_{D_6}(y) &= y^6-30y^5+235y^4-780y^3+1260y^2-980y+294,\\
\chi^*_{D_7}(y) &= y^7-42y^6+456y^5-2135y^4+5110y^3-6552y^2+4284y-1122,\\
\chi^*_{E_6}(y) &= y^6-36y^5+300y^4-1035y^3+1720y^2-1368y+418,\\
\chi^*_{E_7}(y) &= y^7-63y^6+777y^5-3927y^4+9933y^3-13299y^2+9009y-2431,\\
\chi^*_{E_8}(y) &= y^8-120y^7+2135y^6-15120y^5\\
&\qquad\qquad +54327y^4-108360y^3+121555y^2-71760y+17342.
\end{aligned}
\tag{5.2}
$$

6. The M-triangle of generalised non-crossing partitions of type E_7. We now implement the programme outlined in Sections 4 and 5 to compute the decomposition numbers for E_7 and, thus, via (3.1) and (5.2), the M-triangle of the m-divisible non-crossing partitions of type E_7.

To begin with, we have to compute the decomposition numbers for types of smaller ranks, that is, of ranks ≤ 6. The decomposition numbers for the irreducible types of rank ≤ 6 that we need, for A_1, A_2, A_3, A_4, A_5, A_6, D_4, D_5, D_6, and E_6, are given in the Appendix. Those for the reducible types of rank ≤ 6 then can be computed by using Proposition 7.[2] Then we use (4.1) and (4.2) to express all the decomposition numbers in terms of full rank decomposition numbers $N_\Phi(T_1,T_2,\dots,T_d)$ (that is, those with $\operatorname{rk}\Phi=\operatorname{rk}T_1+\operatorname{rk}T_2+\cdots+\operatorname{rk}T_d$), in which the types T_i are ordered (for example, according to lexicographic order). Subsequently we produce the equations which one obtains

[2]Although theoretically totally sound, this is not the most convenient way to do it: the actual way we computed these was to take the setup for an irreducible root system of the same rank following the programme outlined in Section 4, to change the values of the special decomposition numbers given by Propositions 4 and 6, and to solve the new system of linear equations.

from (4.3) (using the earlier computed decomposition numbers of smaller rank) and from comparing coefficients of $m^i z^j$, $i,j \in \{0,1,\dots,7\}$, on both sides of (4.7). Finally, we use Proposition 4 to determine $N_{E_7}(E_7)$, $N_{E_7}(A_1)$ and $N_{E_7}(A_1,A_1,A_1,A_1,A_1,A_1,A_1)$, Proposition 6 to compute the numbers $N_{E_7}(T,A_1)$ with $\operatorname{rk} T=6$, and we use the last assertion in Proposition 2 for the type A_1^5 (which does not occur as the type of a sub-diagram of the Dynkin diagram of E_7). To wit, the latter leads to the following special values of decomposition numbers:

$$\begin{gathered}
N_{E_7}(E_7)=1,\\
N_{E_7}(A_1)=63,\\
N_{E_7}(A_1,A_1,A_1,A_1,A_1,A_1,A_1)=1062882,\\
N_{E_7}(E_6,A_1)=N_{E_7}(D_6,A_1)=N_{E_7}(A_6,A_1)=N_{E_7}(A_1*A_5,A_1)\\
=N_{E_7}(A_1*D_5,A_1)=N_{E_7}(A_2*A_4,A_1)=N_{E_7}(A_1*A_2*A_3,A_1)=9,\\
N_{E_7}(A_2*D_4,A_1)=N_{E_7}(A_3^2,A_1)=N_{E_7}(A_1^2*D_4,A_1)=N_{E_7}(A_1^2*A_4,A_1)\\
=N_{E_7}(A_1^3*A_3,A_1)=N_{E_7}(A_2^3,A_1)=N_{E_7}(A_1^2*A_2^2,A_1)=N_{E_7}(A_1^4*A_2,A_1)\\
=N_{E_7}(A_1^6,A_1)=N_{E_7}(A_1^5,A_2)=N_{E_7}(A_1^5,A_1^2)=N_{E_7}(A_1^5,A_1,A_1)=0.
\end{gathered}$$

We now let *Mathematica 5.2* solve the described system of linear equations for the full rank decomposition numbers.[3] Although this is a system of more than 200 equations with 115 variables, the solution space is two-dimensional. *Mathematica* expresses all the variables in terms of $X=N_{E_7}(A_1^4,A_1^3)$ and $Y=N_{E_7}(A_1^2*A_2,A_1^3)$. For the subsequent considerations, we work with the following selection of relations output by *Mathematica*:

$$N_{E_7}(A_5,A_2)=\frac{1272}{25}-\frac{58}{75}X-\frac{58}{225}Y, \tag{6.1}$$

$$N_{E_7}(A_4,A_3)=\frac{594}{25}+\frac{18}{25}X+\frac{11}{25}Y, \tag{6.2}$$

$$N_{E_7}(D_4,A_3)=\frac{126}{5}-\frac{2}{5}X-\frac{7}{30}Y, \tag{6.3}$$

$$N_{E_7}(A_1^3*A_2,A_1^2)=\frac{423}{5}-\frac{14}{5}X-\frac{14}{15}Y, \tag{6.4}$$

$$N_{E_7}(A_1^3*A_2,A_2)=-\frac{192}{5}+\frac{28}{15}X+\frac{28}{45}Y. \tag{6.5}$$

Relation (6.1) implies

$$X-8Y\equiv 2 \pmod{25}, \tag{6.6}$$

Relation (6.2) implies

$$18X+11Y\equiv 6 \pmod{25}, \tag{6.7}$$

and Relation (6.3) implies

$$Y\equiv 0 \pmod{6}. \tag{6.8}$$

[3]The *Mathematica* input is available at `http://www.mat.univie.ac.at/~kratt/artikel/cluster2.html`.

Solving (6.6), (6.7) and (6.8) for X and Y yields

$$X = 25x + 15y + 4,$$
$$Y = 30y - 6,$$

for some integers x and y with $y > 0$. If we substitute this in (6.4) and (6.5), then we obtain

$$N_{E_7}(A_1^3 * A_2, A_1^2) = 79 - 70(x + y),$$
$$N_{E_7}(A_1^3 * A_2, A_2) = \frac{4}{3}(35(x + y) - 26).$$

Since the decomposition numbers must be non-negative integers, we conclude that $x+y = 1$. Thus, we have $X = 29-10y$ and $Y = 30y-6$, with $y = 1$ or $y = 2$. To decide which of the two values is the true value, we appeal to Lemma 5. Given a decomposition $c = c_1c_2$, where c_1 is of type A_1^4 and c_2 is of type A_1^3, the entire orbit $\mathcal{O} = \{(c^k c_1 c^{-k}, c^k c_2 c^{-k}) : k \in \mathbb{Z}\}$ consists of pairs (x, y) with $c = xy$, where x is of type A_1^4 and y is of type A_1^3. Since Lemma 5 in the case of E_7 says that the orbits of reflections have all size 9, the size of $\mathcal{O}$ must be a divisor of 9. We claim that it cannot be 1. For, in that case c_1 and c_2 commute with the Coxeter element c. By a result of Carter [10, Prop. 30], the centralizer of c is known to be the cyclic group (of order $h = 18$) generated by c. Hence, c_1 and c_2 would have to be powers of c. Since c_1 is of type A_1^4 and c_2 is of type A_1^3, both c_1 and c_2 are of order 2. However, the only element of order 2 in the cyclic group generated by c is c^9, whence $c^9 = c_1 = c_2$, which is absurd. The conclusion is that 3 divides the orbit $\mathcal{O}$. Thus, 3 must divide $X = N_{E_7}(A_1^4, A_1^3)$. Out of the two possible values for y, we must therefore choose $y = 2$, and so $X = 9$ and $Y = N_{E_7}(A_1^2 * A_2, A_1^3) = 54$.

Now we substitute these values in the expressions in terms of X and Y found by *Mathematica* for the other full rank decomposition numbers. The result is that $N_{E_7}(E_7) = 1$, $N_{E_7}(E_6, A_1) = 9$, $N_{E_7}(D_6, A_1) = 9$, $N_{E_7}(A_6, A_1) = 9$, $N_{E_7}(A_1 * D_5, A_1) = 9$, $N_{E_7}(A_1 * A_5, A_1) = 9$, $N_{E_7}(A_2 * D_4, A_1) = 0$, $N_{E_7}(A_2 * A_4, A_1) = 9$, $N_{E_7}(A_1^2 * D_4, A_1) = 0$, $N_{E_7}(A_1^2 * A_4, A_1) = 0$, $N_{E_7}(A_3^2, A_1) = 0$, $N_{E_7}(A_1 * A_2 * A_3, A_1) = 9$, $N_{E_7}(A_1^3 * A_3, A_1) = 0$, $N_{E_7}(A_2^3, A_1) = 0$, $N_{E_7}(A_1^2 * A_2^2, A_1) = 0$, $N_{E_7}(A_1^4 * A_2, A_1) = 0$, $N_{E_7}(A_1^6, A_1) = 0$, $N_{E_7}(D_5, A_2) = 18$, $N_{E_7}(A_5, A_2) = 30$, $N_{E_7}(A_1 * A_4, A_2) = 54$, $N_{E_7}(A_1 * D_4, A_2) = 9$, $N_{E_7}(A_2 * A_3, A_2) = 36$, $N_{E_7}(A_1^2 * A_3, A_2) = 36$, $N_{E_7}(A_1 * A_2^2, A_2) = 36$, $N_{E_7}(A_1^3 * A_2, A_2) = 12$, $N_{E_7}(A_1^5, A_2) = 0$, $N_{E_7}(D_5, A_1^2) = 54$, $N_{E_7}(A_5, A_1^2) = 63$, $N_{E_7}(A_1 * D_4, A_1^2) = 27$, $N_{E_7}(A_1 * A_4, A_1^2) = 81$, $N_{E_7}(A_2 * A_3, A_1^2) = 27$, $N_{E_7}(A_1^2 * A_3, A_1^2) = 27$, $N_{E_7}(A_1 * A_2^2, A_1^2) = 27$, $N_{E_7}(A_1^3 * A_2, A_1^2) = 9$, $N_{E_7}(A_1^5, A_1^2) = 0$, $N_{E_7}(D_5, A_1, A_1) = 162$, $N_{E_7}(A_5, A_1, A_1) = 216$, $N_{E_7}(A_1 * D_4, A_1, A_1) = 81$, $N_{E_7}(A_1 * A_4, A_1, A_1) = 324$, $N_{E_7}(A_2 * A_3, A_1, A_1) = 162$, $N_{E_7}(A_1^2 * A_3, A_1, A_1) = 162$, $N_{E_7}(A_1 * A_2^2, A_1, A_1) = 162$, $N_{E_7}(A_1^3 * A_2, A_1, A_1) = 54$, $N_{E_7}(A_1^5, A_1, A_1) = 0$, $N_{E_7}(D_4, A_3) = 9$, $N_{E_7}(A_4, A_3) = 54$, $N_{E_7}(A_1 * A_3, A_3) = 135$, $N_{E_7}(A_2^2, A_3) = 54$, $N_{E_7}(A_1^2 * A_2, A_3) = 162$, $N_{E_7}(A_1^4, A_3) = 27$, $N_{E_7}(D_4, A_1 * A_2) = 45$, $N_{E_7}(A_4, A_1 * A_2) = 162$, $N_{E_7}(A_1 * A_3, A_1 * A_2) = 243$, $N_{E_7}(A_2^2, A_1 * A_2) = 54$, $N_{E_7}(A_1^2 * A_2, A_1 * A_2) = 162$, $N_{E_7}(A_1^4, A_1 * A_2) = 27$, $N_{E_7}(D_4, A_1^3) = 30$, $N_{E_7}(A_4, A_1^3) = 99$, $N_{E_7}(A_1 * A_3, A_1^3) = 126$, $N_{E_7}(A_2^2, A_1^3) = 18$, $N_{E_7}(A_1^2 * A_2, A_1^3) = 54$, $N_{E_7}(A_1^4, A_1^3) = 9$, $N_{E_7}(D_4, A_2, A_1) = 81$, $N_{E_7}(A_4, A_2, A_1) = 378$, $N_{E_7}(A_1 * A_3, A_2, A_1) = 783$, $N_{E_7}(A_2^2, A_2,$

$A_1) = 270$, $N_{E_7}(A_1^2 * A_2, A_2, A_1) = 810$, $N_{E_7}(A_1^4, A_2, A_1) = 135$, $N_{E_7}(D_4, A_1^2, A_1) = 243$, $N_{E_7}(A_4, A_1^2, A_1) = 891$, $N_{E_7}(A_1 * A_3, A_1^2, A_1) = 1377$, $N_{E_7}(A_2^2, A_1^2, A_1) = 324$, $N_{E_7}(A_1^2 * A_2, A_1^2, A_1) = 972$, $N_{E_7}(A_1^4, A_1^2, A_1) = 162$, $N_{E_7}(D_4, A_1, A_1, A_1) = 729$, $N_{E_7}(A_4, A_1, A_1, A_1) = 2916$, $N_{E_7}(A_1 * A_3, A_1, A_1, A_1) = 5103$, $N_{E_7}(A_2^2, A_1, A_1, A_1) = 1458$, $N_{E_7}(A_1^2 * A_2, A_1, A_1, A_1) = 4374$, $N_{E_7}(A_1^4, A_1, A_1, A_1) = 729$, $N_{E_7}(A_3, A_3, A_1) = 486$, $N_{E_7}(A_3, A_1 * A_2, A_1) = 1458$, $N_{E_7}(A_3, A_1^3, A_1) = 891$, $N_{E_7}(A_1 * A_2, A_1 * A_2, A_1) = 2430$, $N_{E_7}(A_1 * A_2, A_1^3, A_1) = 1215$, $N_{E_7}(A_1^3, A_1^3, A_1) = 540$, $N_{E_7}(A_3, A_2, A_2) = 432$, $N_{E_7}(A_1 * A_2, A_2, A_2) = 1188$, $N_{E_7}(A_1^3, A_2, A_2) = 711$, $N_{E_7}(A_3, A_2, A_1^2) = 1053$, $N_{E_7}(A_1 * A_2, A_2, A_1^2) = 2349$, $N_{E_7}(A_1^3, A_2, A_1^2) = 1323$, $N_{E_7}(A_3, A_1^2, A_1^2) = 2430$, $N_{E_7}(A_1 * A_2, A_1^2, A_1^2) = 3402$, $N_{E_7}(A_1^3, A_1^2, A_1^2) = 1539$, $N_{E_7}(A_3, A_2, A_1, A_1) = 3402$, $N_{E_7}(A_1 * A_2, A_2, A_1, A_1) = 8262$, $N_{E_7}(A_1^3, A_2, A_1, A_1) = 4779$, $N_{E_7}(A_3, A_1^2, A_1, A_1) = 8019$, $N_{E_7}(A_1 * A_2, A_1^2, A_1, A_1) = 13851$, $N_{E_7}(A_1^3, A_1^2, A_1, A_1) = 7047$, $N_{E_7}(A_3, A_1, A_1, A_1, A_1) = 26244$, $N_{E_7}(A_1 * A_2, A_1, A_1, A_1, A_1) = 52488$, $N_{E_7}(A_1^3, A_1, A_1, A_1, A_1) = 28431$, $N_{E_7}(A_2, A_2, A_2, A_1) = 2916$, $N_{E_7}(A_2, A_2, A_1^2, A_1) = 6561$, $N_{E_7}(A_2, A_1^2, A_1^2, A_1) = 13122$, $N_{E_7}(A_1^2, A_1^2, A_1^2, A_1) = 19683$, $N_{E_7}(A_2, A_2, A_1, A_1, A_1) = 21870$, $N_{E_7}(A_2, A_1^2, A_1, A_1, A_1) = 45927$, $N_{E_7}(A_1^2, A_1^2, A_1, A_1, A_1) = 78732$, $N_{E_7}(A_2, A_1, A_1, A_1, A_1, A_1) = 157464$, $N_{E_7}(A_1^2, A_1, A_1, A_1, A_1, A_1) = 295245$, $N_{E_7}(A_1, A_1, A_1, A_1, A_1, A_1, A_1) = 1062882$, plus the assignments implied by (4.1) and (4.2), all other numbers $N_{E_7}(T_1, \dots, T_d)$ being zero.

Finally, we substitute the values found for the decomposition numbers and the formulae for the characteristic polynomials from (5.2) in (3.1), and we obtain

$$\begin{aligned}
(M_{E_7}^m)^*(x,y) = {}& \frac{1}{280} m(9m-2)(9m-4)(9m-5)(9m-8)(3m-1)(3m-2)x^7y^7 \\
&- \frac{9}{40} m^2(9m-2)(9m-5)(3m-1)(3m-2)(9m-4)x^7y^6 \\
&+ \frac{3}{40} m^2(9m-2)(9m-4)(3m-1)(243m^2-81m-14)x^7y^5 \\
&- \frac{3}{8} m^2(3m-1)(9m-2)(9m+2)(81m^2-9m-4)x^7y^4 \\
&+ \frac{3}{8} m^2(9m-2)(3m+1)(9m+2)(81m^2+9m-4)x^7y^3 \\
&- \frac{3}{40} m^2(3m+1)(9m+2)(9m+4)(243m^2+81m-14)x^7y^2 \\
&+ \frac{9}{40} m^2(3m+1)(3m+2)(9m+2)(9m+4)(9m+5)x^7y \\
&- \frac{1}{280} m(3m+1)(3m+2)(9m+2)(9m+4)(9m+5)(9m+8)x^7 \\
&+ \frac{9}{40} m(9m-2)(9m-5)(3m-1)(3m-2)(9m-4)x^6y^6 \\
&- \frac{27}{20} m^2(27m-13)(9m-2)(9m-4)(3m-1)x^6y^5 \\
&+ \frac{27}{8} m^2(3m-1)(9m-2)(243m^2-45m-10)x^6y^4 - \frac{27}{2} m^2(9m-2)(9m+2)(27m^2-1)x^6y^3 \\
&+ \frac{27}{8} m^2(3m+1)(9m+2)(243m^2+45m-10)x^6y^2 - \frac{27}{20} m^2(3m+1)(9m+2)(9m+4)(27m+13)x^6y \\
&+ \frac{9}{40} m(3m+1)(3m+2)(9m+2)(9m+4)(9m+5)x^6 + \frac{3}{40} m(207m-103)(9m-2)(9m-4)(3m-1)x^5y^5 \\
&- \frac{27}{8} m^2(207m-71)(3m-1)(9m-2)x^5y^4 + \frac{9}{4} m^2(9m-2)(1863m^2-144m-55)x^5y^3
\end{aligned}$$

$$
\begin{aligned}
&- \frac{9}{4} m^2(9m+2)(1863m^2+144m-55)x^5y^2 + \frac{27}{8} m^2(3m+1)(9m+2)(207m+71)x^5y \\
&- \frac{3}{40} m(3m+1)(9m+2)(9m+4)(207m+103)x^5 + \frac{21}{8} m(63m-23)(3m-1)(9m-2)x^4y^4 \\
&- \frac{189}{2} m^2(21m-5)(9m-2)x^4y^3 + \frac{63}{4} m^2(1701m^2-37)x^4y^2 - \frac{189}{2} m^2(9m+2)(21m+5)x^4y \\
&+ \frac{21}{8} m(3m+1)(9m+2)(63m+23)x^4 + \frac{21}{2} m(27m-7)(9m-2)x^3y^3 - \frac{189}{2} m^2(81m-13)x^3y^2 \\
&+ \frac{189}{2} m^2(81m+13)x^3y - \frac{21}{2} m(9m+2)(27m+7)x^3 + \frac{21}{2} m(63m-11)x^2y^2 \\
&- 1323m^2x^2y + \frac{21}{2} m(63m+11)x^2 + 63mxy - 63mx + 1. \qquad (6.9)
\end{aligned}
$$

7. The M-triangle of generalised non-crossing partitions of type E_8. In this section we implement the programme outlined in Sections 4 and 5 to compute the decomposition numbers for E_8 and, thus, via (3.1) and (5.2), the M-triangle of the m-divisible non-crossing partitions of type E_8.

Again, to begin with, we have to compute the decomposition numbers for types of smaller ranks, that is, of ranks ≤ 7. The decomposition numbers for the irreducible types of rank ≤ 7 that we need, for A_1, A_2, A_3, A_4, A_5, A_6, A_7, D_4, D_5, D_6, D_7, E_6, and E_7, are given in the Appendix, respectively in Section 6. Those for the reducible types of rank ≤ 7 then can be computed by using Proposition 7.[4] Then we use (4.1) and (4.2) to express all the decomposition numbers in terms of full rank decomposition numbers $N_\Phi(T_1, T_2, \dots, T_d)$ (that is, those with $\operatorname{rk}\Phi = \operatorname{rk}T_1 + \operatorname{rk}T_1 + \cdots + \operatorname{rk}T_d$), in which the types T_i are ordered (for example, according to lexicographic order). Subsequently we produce the equations which one obtains from (4.3) (using the earlier computed decomposition numbers of smaller rank) and from comparing coefficients of $m^i z^j$, $i, j \in \{0, 1, \dots, 8\}$, on both sides of (4.7). Finally, we use Proposition 4 to determine $N_{E_8}(E_8)$, $N_{E_8}(A_1)$ and $N_{E_8}(A_1, A_1, A_1, A_1, A_1, A_1, A_1, A_1)$, Proposition 6 to compute the numbers $N_{E_8}(T, A_1)$ with $\operatorname{rk}T = 7$, and we use the last assertion in Proposition 2 for the types A_1^6, A_1^5, $A_1^3 * A_3$, A_2^3 and $A_1^2 * D_4$ (which do not occur as the type of a sub-diagram of the Dynkin diagram of E_8). To wit, the latter leads to the following special values of decomposition numbers:

$$
\begin{aligned}
N_{E_8}(E_8) &= 1, \\
N_{E_8}(A_1) &= 120, \\
N_{E_8}(A_1, A_1, A_1, A_1, A_1, A_1, A_1, A_1) &= 37968750, \\
N_{E_8}(E_7, A_1) = N_{E_8}(D_7, A_1) = N_{E_8}(A_7, A_1) &= N_{E_8}(A_1 * E_6, A_1) \\
= N_{E_8}(A_1 * A_6, A_1) &= N_{E_8}(A_2 * D_5, A_1) \\
= N_{E_8}(A_3 * A_4, A_1) &= N_{E_8}(A_1 * A_2 * A_4, A_1) = 15,
\end{aligned}
$$

[4]See Footnote 1.

$$\begin{aligned}
N_{E_8}(A_1 * D_6, A_1) &= N_{E_8}(A_2 * A_5, A_1) = N_{E_8}(A_1^2 * D_5, A_1) = N_{E_8}(A_1^2 * A_5, A_1)\\
&= N_{E_8}(A_3 * D_4, A_1) = N_{E_8}(A_1 * A_2 * D_4, A_1) = N_{E_8}(A_1^3 * D_4, A_1)\\
&= N_{E_8}(A_1^3 * A_4, A_1) = N_{E_8}(A_1 * A_3^2, A_1) = N_{E_8}(A_2^2 * A_3, A_1)\\
&= N_{E_8}(A_1^2 * A_2 * A_3, A_1) = N_{E_8}(A_1^4 * A_3, A_1) = N_{E_8}(A_1 * A_2^3, A_1)\\
&= N_{E_8}(A_1^3 * A_2^2, A_1) = N_{E_8}(A_1^5 * A_2, A_1) = N_{E_8}(A_1^7, A_1)\\
&= N_{E_8}(A_1^6, A_2) = N_{E_8}(A_1^6, A_1^2) = N_{E_8}(A_1^6, A_1, A_1) = N_{E_8}(A_1^5, A_3)\\
&= N_{E_8}(A_1^5, A_1 * A_2) = N_{E_8}(A_1^5, A_1^3) = N_{E_8}(A_1^5, A_2, A_1)\\
&= N_{E_8}(A_1^5, A_1^2, A_1) = N_{E_8}(A_1^5, A_1, A_1, A_1) = N_{E_8}(A_1^3 * A_3, A_2)\\
&= N_{E_8}(A_1^3 * A_3, A_1^2) = N_{E_8}(A_1^3 * A_3, A_1, A_1) = N_{E_8}(A_2^3, A_2)\\
&= N_{E_8}(A_2^3, A_1^2) = N_{E_8}(A_1^4 * A_2, A_2) = N_{E_8}(A_1^4 * A_2, A_1^2)\\
&= N_{E_8}(A_1^2 * D_4, A_2) = N_{E_8}(A_1^2 * D_4, A_1^2) = 0.
\end{aligned}$$

Again, we let *Mathematica 5.2* solve the described system of linear equations for the full rank decomposition numbers.[5] Although this is now a system of more than 600 equations with only about 250 variables, the situation here is even worse as the solution space is four-dimensional. *Mathematica* expresses all the variables in terms of $X = N_{E_8}(A_5, A_1 * A_2)$, $Y = N_{E_8}(D_5, A_1 * A_2)$, $N_{E_8}(A_4, A_1 * A_3)$, and $N_{E_8}(D_4, A_4)$. For the subsequent considerations, we work with the following selection of relations output by *Mathematica*:

$$N_{E_8}(A_5, A_3) = \frac{1}{4}(750 - X), \tag{7.1}$$

$$N_{E_8}(D_5, A_3) = \frac{1}{4}(375 - Y), \tag{7.2}$$

$$N_{E_8}(A_5, A_1^3) = \frac{5}{6}(750 - X), \tag{7.3}$$

$$N_{E_8}(D_5, A_1^3) = \frac{5}{6}(375 - Y), \tag{7.4}$$

$$N_{E_8}(A_2 * A_3, A_1 * A_2) = \frac{3}{25}(4125 - 8X + 16Y), \tag{7.5}$$

$$N_{E_8}(A_2 * A_3, A_1^3) = \frac{1}{5}(1125 + 4X - 8Y), \tag{7.6}$$

$$N_{E_8}(A_1 * D_4, A_3) = Y - 150, \tag{7.7}$$

$$N_{E_8}(A_1^2 * A_3, A_1^3) = \frac{1}{5}(49875 - 56X - 138Y), \tag{7.8}$$

$$N_{E_8}(A_1 * A_4, A_1 * A_2) = 2(2295 - 3X - 4Y), \tag{7.9}$$

$$N_{E_8}(A_1^3 * A_2, A_1^3) = \frac{1}{15}(112X + 226Y - 86625). \tag{7.10}$$

There is one further relation which we shall make use of,

$$\begin{aligned}
N_{E_8}(D_4) &= N_{E_8}(D_4, D_4) + N_{E_8}(D_4, A_4) + N_{E_8}(D_4, A_3 * A_1)\\
&\qquad + N_{E_8}(D_4, A_2^2) + N_{E_8}(D_4, A_2 * A_1^2) + N_{E_8}(D_4, A_1^4) \qquad (7.11)\\
&= \frac{27263}{168} - \frac{1}{40}N_{E_8}(A_4, A_1 * A_3) + \frac{1}{40}N_{E_8}(D_4, A_4) + \frac{283}{1500}X + \frac{7957}{15750}Y, \qquad (7.12)
\end{aligned}$$

[5]The *Mathematica* input is available at `http://www.mat.univie.ac.at/~kratt/artikel/cluster2.html`.

where the first line is an instance of (4.2), and where the second line follows from the first upon substituting the relations for the full rank decomposition numbers output by *Mathematica*.

Relations (7.1) and (7.3) imply

$$X \equiv 6 \pmod{12}, \tag{7.13}$$

and Relations (7.2) and (7.4) imply

$$Y \equiv 3 \pmod{12}. \tag{7.14}$$

Next, Relation (7.5) implies

$$X \equiv 2Y \pmod{25}. \tag{7.15}$$

Solving (7.13), (7.14) and (7.15) for X and Y yields

$$X = 300x + 24y + 6, \tag{7.16}$$

$$Y = 12y + 3, \tag{7.17}$$

for some integers x and y with $y \geq 0$. If we substitute this in (7.1), (7.6)–(7.10), then we obtain

$$N_{E_8}(A_5, A_3) = 3(62 - 25x - 2y), \tag{7.18}$$

$$N_{E_8}(A_2 * A_3, A_1^3) = 225 + 240x, \tag{7.19}$$

$$N_{E_8}(A_1 * D_4, A_3) = 12y - 147, \tag{7.20}$$

$$N_{E_8}(A_1^2 * A_3, A_1^3) = 15(655 - 224x - 40y), \tag{7.21}$$

$$N_{E_8}(A_1 * A_4, A_1 * A_2) = 30(151 - 60x - 8y), \tag{7.22}$$

$$N_{E_8}(A_1^3 * A_2, A_1^3) = 5(448x + 72y - 1137). \tag{7.23}$$

From (7.18) we infer $x \leq 2$, while (7.19) implies $x \geq 0$. Furthermore, from (7.20) we infer $y \geq 13$, while (7.21) implies $y \leq 16$. If $x \geq 1$, then (7.22) implies $y \leq 91/8 < 13$, a contradiction. Hence, $x = 0$. In this case, Equation (7.23) implies $y \geq 1137/72 > 15$, and so $y = 16$. Substituting this in (7.16) and (7.17), we obtain $X = N_{E_8}(A_5, A_1 * A_2) = 390$ and $Y = N_{E_8}(D_5, A_1 * A_2) = 195$.

Unfortunately, similar arithmetic and positivity considerations do not suffice to determine the remaining two "free variables," the decomposition numbers $N_{E_8}(A_4, A_1 * A_3)$ and $N_{E_8}(D_4, A_4)$. Instead, by a 12 days *Maple* computation using Stembridge's `coxeter` package, we found that $N_{E_8}(D_4) = 325$ and $N_{E_8}(D_4, A_4) = 15$. If we use these values, together with the already determined values for X and Y, in (7.12), we eventually find that $N_{E_8}(A_4, A_1 * A_3) = 390$.

We now substitute the above values for $N_{E_8}(A_5, A_1 * A_2)$, $N_{E_8}(D_5, A_1 * A_2)$, $N_{E_8}(A_4, A_1 * A_3)$, and $N_{E_8}(D_4, A_4)$ in the expressions found by *Mathematica* for the other full rank decomposition numbers. The result is that $N_{E_8}(E_8) = 1$, $N_{E_8}(E_7, A_1) = 15$, $N_{E_8}(D_7, A_1) = 15$, $N_{E_8}(A_7, A_1) = 15$, $N_{E_8}(A_1 * E_6, A_1) = 15$, $N_{E_8}(A_1 * D_6, A_1) = 0$,

$N_{E_8}(A_1{*}A_6, A_1) = 15$, $N_{E_8}(A_2{*}D_5, A_1) = 15$, $N_{E_8}(A_2{*}A_5, A_1) = 0$,$N_{E_8}(A_1^2{*}D_5, A_1) = 0$, $N_{E_8}(A_1^2 * A_5, A_1) = 0$, $N_{E_8}(A_3 * D_4, A_1) = 0$, $N_{E_8}(A_3 * A_4, A_1) = 15$, $N_{E_8}(A_1 * A_2 * D_4, A_1) = 0$, $N_{E_8}(A_1 * A_2 * A_4, A_1) = 15$, $N_{E_8}(A_1^3 * D_4, A_1) = 0$, $N_{E_8}(A_1^3 * A_4, A_1) = 0$, $N_{E_8}(A_1 * A_3^2, A_1) = 0$, $N_{E_8}(A_2^2 * A_3, A_1) = 0$, $N_{E_8}(A_1^2 * A_2 * A_3, A_1) = 0$, $N_{E_8}(A_1^4 * A_3, A_1) = 0$, $N_{E_8}(A_1 * A_2^3, A_1) = 0$, $N_{E_8}(A_1^3 * A_2^2, A_1) = 0$, $N_{E_8}(A_1^5 * A_2, A_1) = 0$, $N_{E_8}(A_1^7, A_1) = 0$, $N_{E_8}(E_6, A_2) = 20$, $N_{E_8}(D_6, A_2) = 15$, $N_{E_8}(A_6, A_2) = 60$, $N_{E_8}(A_1 * D_5, A_2) = 60$, $N_{E_8}(A_1 * A_5, A_2) = 60$, $N_{E_8}(A_2 * D_4, A_2) = 20$, $N_{E_8}(A_2 * A_4, A_2) = 90$, $N_{E_8}(A_3^2, A_2) = 45$, $N_{E_8}(A_1^2 * D_4, A_2) = 0$, $N_{E_8}(A_1^2 * A_4, A_2) = 90$, $N_{E_8}(A_1 * A_2 * A_3, A_2) = 90$, $N_{E_8}(A_1^3 * A_3, A_2) = 0$, $N_{E_8}(A_2^3, A_2) = 0$, $N_{E_8}(A_1^2 * A_2^2, A_2) = 45$, $N_{E_8}(A_1^4 * A_2, A_2) = 0$, $N_{E_8}(A_1^6, A_2) = 0$, $N_{E_8}(E_6, A_1^2) = 45$, $N_{E_8}(D_6, A_1^2) = 90$, $N_{E_8}(A_6, A_1^2) = 135$, $N_{E_8}(A_1 * D_5, A_1^2) = 135$, $N_{E_8}(A_1 * A_5, A_1^2) = 135$, $N_{E_8}(A_2 * D_4, A_1^2) = 45$, $N_{E_8}(A_2 * A_4, A_1^2) = 90$, $N_{E_8}(A_3^2, A_1^2) = 45$, $N_{E_8}(A_1^2 * D_4, A_1^2) = 0$, $N_{E_8}(A_1^2 * A_4, A_1^2) = 90$, $N_{E_8}(A_1 * A_2 * A_3, A_1^2) = 90$, $N_{E_8}(A_1^3 * A_3, A_1^2) = 0$, $N_{E_8}(A_2^3, A_1^2) = 0$, $N_{E_8}(A_1^2 * A_2^2, A_1^2) = 45$, $N_{E_8}(A_1^4 * A_2, A_1^2) = 0$, $N_{E_8}(A_1^6, A_1^2) = 0$, $N_{E_8}(E_6, A_1, A_1) = 150$, $N_{E_8}(D_6, A_1, A_1) = 225$, $N_{E_8}(A_6, A_1, A_1) = 450$, $N_{E_8}(A_1 * D_5, A_1, A_1) = 450$, $N_{E_8}(A_1 * A_5, A_1, A_1) = 450$, $N_{E_8}(A_2{*}D_4, A_1, A_1) = 150$, $N_{E_8}(A_2{*}A_4, A_1, A_1) = 450$, $N_{E_8}(A_3^2, A_1, A_1) = 225$, $N_{E_8}(A_1^2 * D_4, A_1, A_1) = 0$, $N_{E_8}(A_1^2 * A_4, A_1, A_1) = 450$, $N_{E_8}(A_1 * A_2 * A_3, A_1, A_1) = 450$, $N_{E_8}(A_1^3 * A_3, A_1, A_1) = 0$, $N_{E_8}(A_2^3, A_1, A_1) = 0$, $N_{E_8}(A_1^2{*}A_2^2, A_1, A_1) = 225$, $N_{E_8}(A_1^4{*}A_2, A_1, A_1) = 0$, $N_{E_8}(A_1^6, A_1, A_1) = 0$, $N_{E_8}(D_5, A_3) = 45$, $N_{E_8}(A_5, A_3) = 90$, $N_{E_8}(A_1 * A_4, A_3) = 315$, $N_{E_8}(A_1 * D_4, A_3) = 45$, $N_{E_8}(A_2 * A_3, A_3) = 270$, $N_{E_8}(A_1^2 * A_3, A_3) = 270$, $N_{E_8}(A_1 * A_2^2, A_3) = 225$, $N_{E_8}(A_1^3 * A_2, A_3) = 225$, $N_{E_8}(A_1^5, A_3) = 0$, $N_{E_8}(D_5, A_1 * A_2) = 195$, $N_{E_8}(A_5, A_1 * A_2) = 390$, $N_{E_8}(A_1 * A_4, A_1 * A_2) = 690$, $N_{E_8}(A_1 * D_4, A_1 * A_2) = 195$, $N_{E_8}(A_2 * A_3, A_1 * A_2) = 495$, $N_{E_8}(A_1^2 * A_3, A_1 * A_2) = 495$, $N_{E_8}(A_1 * A_2^2, A_1 * A_2) = 300$, $N_{E_8}(A_1^3 * A_2, A_1 * A_2) = 300$, $N_{E_8}(A_1^5, A_1 * A_2) = 0$, $N_{E_8}(D_5, A_1^3) = 150$, $N_{E_8}(A_5, A_1^3) = 300$, $N_{E_8}(A_1 * A_4, A_1^3) = 375$, $N_{E_8}(A_1 * D_4, A_1^3) = 150$, $N_{E_8}(A_2 * A_3, A_1^3) = 225$, $N_{E_8}(A_1^2 * A_3, A_1^3) = 225$, $N_{E_8}(A_1 * A_2^2, A_1^3) = 75$, $N_{E_8}(A_1^3 * A_2, A_1^3) = 75$, $N_{E_8}(A_1^5, A_1^3) = 0$, $N_{E_8}(D_5, A_2, A_1) = 375$, $N_{E_8}(A_5, A_2, A_1) = 750$, $N_{E_8}(A_1 * A_4, A_2, A_1) = 1950$, $N_{E_8}(A_1 * D_4, A_2, A_1) = 375$, $N_{E_8}(A_2 * A_3, A_2, A_1) = 1575$, $N_{E_8}(A_1^2 * A_3, A_2, A_1) = 1575$, $N_{E_8}(A_1 * A_2^2, A_2, A_1) = 1200$, $N_{E_8}(A_1^3{*}A_2, A_2, A_1) = 1200$, $N_{E_8}(A_1^5, A_2, A_1) = 0$, $N_{E_8}(D_5, A_1^2, A_1) = 1125$, $N_{E_8}(A_5, A_1^2, A_1) = 2250$, $N_{E_8}(A_1{*}A_4, A_1^2, A_1) = 3825$, $N_{E_8}(A_1{*}D_4, A_1^2, A_1) = 1125$, $N_{E_8}(A_2 * A_3, A_1^2, A_1) = 2700$, $N_{E_8}(A_1^2 * A_3, A_1^2, A_1) = 2700$, $N_{E_8}(A_1 * A_2^2, A_1^2, A_1) = 1575$, $N_{E_8}(A_1^3 * A_2, A_1^2, A_1) = 1575$, $N_{E_8}(A_1^5, A_1^2, A_1) = 0$, $N_{E_8}(D_5, A_1, A_1, A_1) = 3375$, $N_{E_8}(A_5, A_1, A_1, A_1) = 6750$, $N_{E_8}(A_1{*}A_4, A_1, A_1, A_1) = 13500$, $N_{E_8}(A_1{*}D_4, A_1, A_1, A_1) = 3375$, $N_{E_8}(A_2 * A_3, A_1, A_1, A_1) = 10125$, $N_{E_8}(A_1^2 * A_3, A_1, A_1, A_1) = 10125$, $N_{E_8}(A_1 * A_2^2, A_1, A_1, A_1) = 6750$, $N_{E_8}(A_1^3 * A_2, A_1, A_1, A_1) = 6750$, $N_{E_8}(A_1^5, A_1, A_1, A_1) = 0$, $N_{E_8}(D_4, D_4) = 5$, $N_{E_8}(D_4, A_4) = 15$, $N_{E_8}(A_4, A_4) = 138$, $N_{E_8}(D_4, A_1 * A_3) = 105$, $N_{E_8}(A_4, A_1{*}A_3) = 390$, $N_{E_8}(A_1{*}A_3, A_1{*}A_3) = 1155$, $N_{E_8}(D_4, A_2^2) = 35$, $N_{E_8}(A_4, A_2^2) = 180$, $N_{E_8}(A_1 * A_3, A_2^2) = 360$, $N_{E_8}(A_2^2, A_2^2) = 95$, $N_{E_8}(D_4, A_1^2 * A_2) = 135$, $N_{E_8}(A_4, A_1^2 * A_2) = 630$, $N_{E_8}(A_1{*}A_3, A_1^2{*}A_2) = 1035$, $N_{E_8}(A_2^2, A_1^2{*}A_2) = 270$, $N_{E_8}(A_1^2{*}A_2, A_1^2{*}A_2) = 495$, $N_{E_8}(D_4, A_1^4) = 30$, $N_{E_8}(A_4, A_1^4) = 165$, $N_{E_8}(A_1 * A_3, A_1^4) = 255$, $N_{E_8}(A_2^2, A_1^4) = 60$, $N_{E_8}(A_1^2 * A_2, A_1^4) = 135$, $N_{E_8}(A_1^4, A_1^4) = 30$, $N_{E_8}(D_4, A_3, A_1) = 225$, $N_{E_8}(A_4, A_3, A_1) = 1215$, $N_{E_8}(A_1{*}A_3, A_3, A_1) = 4050$, $N_{E_8}(A_2^2, A_3, A_1) = 1575$, $N_{E_8}(A_1^2{*}A_2, A_3, A_1) = 5400$, $N_{E_8}(A_1^4, A_3, A_1) = 1350$, $N_{E_8}(D_4, A_1 * A_2, A_1) = 975$, $N_{E_8}(A_4, A_1 * A_2, A_1) = 4590$, $N_{E_8}(A_1{*}A_3, A_1{*}A_2, A_1) = 10800$, $N_{E_8}(A_2^2, A_1{*}A_2, A_1) = 3450$, $N_{E_8}(A_1^2{*}A_2, A_1{*}A_2, A_1) =$

9900, $N_{E_8}(A_1^4, A_1 * A_2, A_1) = 2475$, $N_{E_8}(D_4, A_1^3, A_1) = 750$, $N_{E_8}(A_4, A_1^3, A_1) = 3375$, $N_{E_8}(A_1 * A_3, A_1^3, A_1) = 6750$, $N_{E_8}(A_2^2, A_1^3, A_1) = 1875$, $N_{E_8}(A_1^2 * A_2, A_1^3, A_1) = 4500$, $N_{E_8}(A_1^4, A_1^3, A_1) = 1125$, $N_{E_8}(D_4, A_2, A_2) = 175$, $N_{E_8}(A_4, A_2, A_2) = 1140$, $N_{E_8}(A_1 * A_3, A_2, A_2) = 3300$, $N_{E_8}(A_2^2, A_2, A_2) = 1300$, $N_{E_8}(A_1^2 * A_2, A_2, A_2) = 4500$, $N_{E_8}(A_1^4, A_2, A_2) = 1125$, $N_{E_8}(D_4, A_2, A_1^2) = 675$, $N_{E_8}(A_4, A_2, A_1^2) = 3015$, $N_{E_8}(A_1 * A_3, A_2, A_1^2) = 8550$, $N_{E_8}(A_2^2, A_2, A_1^2) = 2925$, $N_{E_8}(A_1^2 * A_2, A_2, A_1^2) = 9000$, $N_{E_8}(A_1^4, A_2, A_1^2) = 2250$, $N_{E_8}(D_4, A_1^2, A_1^2) = 1800$, $N_{E_8}(A_4, A_1^2, A_1^2) = 8640$, $N_{E_8}(A_1 * A_3, A_1^2, A_1^2) = 17550$, $N_{E_8}(A_2^2, A_1^2, A_1^2) = 5175$, $N_{E_8}(A_1^2 * A_2, A_1^2, A_1^2) = 13500$, $N_{E_8}(A_1^4, A_1^2, A_1^2) = 3375$, $N_{E_8}(D_4, A_2, A_1, A_1) = 1875$, $N_{E_8}(A_4, A_2, A_1, A_1) = 9450$, $N_{E_8}(A_1 * A_3, A_2, A_1, A_1) = 27000$, $N_{E_8}(A_2^2, A_2, A_1, A_1) = 9750$, $N_{E_8}(A_1^2 * A_2, A_2, A_1, A_1) = 31500$, $N_{E_8}(A_1^4, A_2, A_1, A_1) = 7875$, $N_{E_8}(D_4, A_1^2, A_1, A_1) = 5625$, $N_{E_8}(A_4, A_1^2, A_1, A_1) = 26325$, $N_{E_8}(A_1 * A_3, A_1^2, A_1, A_1) = 60750$, $N_{E_8}(A_2^2, A_1^2, A_1, A_1) = 19125$, $N_{E_8}(A_1^2 * A_2, A_1^2, A_1, A_1) = 54000$, $N_{E_8}(A_1^4, A_1^2, A_1, A_1) = 13500$, $N_{E_8}(D_4, A_1, A_1, A_1, A_1) = 16875$, $N_{E_8}(A_4, A_1, A_1, A_1, A_1) = 81000$, $N_{E_8}(A_1 * A_3, A_1, A_1, A_1, A_1) = 202500$, $N_{E_8}(A_2^2, A_1, A_1, A_1, A_1) = 67500$, $N_{E_8}(A_1^2 * A_2, A_1, A_1, A_1, A_1) = 202500$, $N_{E_8}(A_1^4, A_1, A_1, A_1, A_1) = 50625$, $N_{E_8}(A_3, A_3, A_2) = 1350$, $N_{E_8}(A_3, A_1 * A_2, A_2) = 5175$, $N_{E_8}(A_3, A_1^3, A_2) = 3825$, $N_{E_8}(A_1 * A_2, A_1 * A_2, A_2) = 15000$, $N_{E_8}(A_1 * A_2, A_1^3, A_2) = 9825$, $N_{E_8}(A_1^3, A_1^3, A_2) = 6000$, $N_{E_8}(A_3, A_3, A_1^2) = 4050$, $N_{E_8}(A_3, A_1 * A_2, A_1^2) = 13500$, $N_{E_8}(A_3, A_1^3, A_1^2) = 9450$, $N_{E_8}(A_1 * A_2, A_1 * A_2, A_1^2) = 30825$, $N_{E_8}(A_1 * A_2, A_1^3, A_1^2) = 17325$, $N_{E_8}(A_1^3, A_1^3, A_1^2) = 7875$, $N_{E_8}(A_3, A_3, A_1, A_1) = 12150$, $N_{E_8}(A_3, A_1 * A_2, A_1, A_1) = 42525$, $N_{E_8}(A_3, A_1^3, A_1, A_1) = 30375$, $N_{E_8}(A_1 * A_2, A_1 * A_2, A_1, A_1) = 106650$, $N_{E_8}(A_1 * A_2, A_1^3, A_1, A_1) = 64125$, $N_{E_8}(A_1^3, A_1^3, A_1, A_1) = 33750$, $N_{E_8}(A_3, A_2, A_2, A_1) = 10575$, $N_{E_8}(A_3, A_2, A_1^2, A_1) = 29700$, $N_{E_8}(A_3, A_1^2, A_1^2, A_1) = 76950$, $N_{E_8}(A_1 * A_2, A_2, A_2, A_1) = 35700$, $N_{E_8}(A_1 * A_2, A_2, A_1^2, A_1) = 84825$, $N_{E_8}(A_1 * A_2, A_1^2, A_1^2, A_1) = 171450$, $N_{E_8}(A_1^3, A_2, A_2, A_1) = 25125$, $N_{E_8}(A_1^3, A_2, A_1^2, A_1) = 55125$, $N_{E_8}(A_1^3, A_1^2, A_1^2, A_1) = 94500$, $N_{E_8}(A_3, A_2, A_1, A_1, A_1) = 91125$, $N_{E_8}(A_3, A_1^2, A_1, A_1, A_1) = 243000$, $N_{E_8}(A_1 * A_2, A_2, A_1, A_1, A_1) = 276750$, $N_{E_8}(A_1 * A_2, A_1^2, A_1, A_1, A_1) = 597375$, $N_{E_8}(A_1^3, A_2, A_1, A_1, A_1) = 185625$, $N_{E_8}(A_1^3, A_1^2, A_1, A_1, A_1) = 354375$, $N_{E_8}(A_3, A_1, A_1, A_1, A_1, A_1) = 759375$, $N_{E_8}(A_1 * A_2, A_1, A_1, A_1, A_1, A_1) = 2025000$, $N_{E_8}(A_1^3, A_1, A_1, A_1, A_1, A_1) = 1265625$, $N_{E_8}(A_2, A_2, A_2, A_2) = 9350$, $N_{E_8}(A_2, A_2, A_2, A_1^2) = 24975$, $N_{E_8}(A_2, A_2, A_1^2, A_1^2) = 64350$, $N_{E_8}(A_2, A_1^2, A_1^2, A_1^2) = 143100$, $N_{E_8}(A_1^2, A_1^2, A_1^2, A_1^2) = 261225$, $N_{E_8}(A_2, A_2, A_2, A_1, A_1) = 78000$, $N_{E_8}(A_2, A_2, A_1^2, A_1, A_1) = 203625$, $N_{E_8}(A_2, A_1^2, A_1^2, A_1, A_1) = 479250$, $N_{E_8}(A_1^2, A_1^2, A_1^2, A_1, A_1) = 951750$, $N_{E_8}(A_2, A_2, A_1, A_1, A_1, A_1) = 641250$, $N_{E_8}(A_2, A_1^2, A_1, A_1, A_1, A_1) = 1569375$, $N_{E_8}(A_1^2, A_1^2, A_1, A_1, A_1, A_1) = 3341250$, $N_{E_8}(A_2, A_1, A_1, A_1, A_1, A_1, A_1) = 5062500$, $N_{E_8}(A_1^2, A_1, A_1, A_1, A_1, A_1, A_1) = 11390625$, $N_{E_8}(A_1, A_1, A_1, A_1, A_1, A_1, A_1, A_1) = 37968750$, plus the assignments implied by (4.1) and (4.2), all other numbers $N_{E_8}(T_1, \dots, T_d)$ being zero.

Finally, we substitute the values found for the decomposition numbers and the formulae for the characteristic polynomials from (5.2) in (3.1), and we obtain

$$\begin{aligned}
(M_{E_8}^m)^*(x,y) &= \frac{1}{1344} m(15m-8)(15m-11)(15m-14)(5m-1)(5m-2)(3m-1)(5m-3)x^8y^8 \\
&\quad - \frac{5}{56} m^2(15m-8)(15m-11)(5m-1)(5m-2)(5m-3)(3m-1)x^8y^7 \\
&\quad + \frac{5}{48} m^2(15m-8)(5m-1)(3m-1)(5m-2)(225m^2-90m-13)x^8y^6
\end{aligned}$$

$$
\begin{aligned}
&-\frac{15}{8}m^2(5m-1)(5m-2)(3m-1)(5m+1)(75m^2-15m-4)x^8y^5\\
&+\frac{1}{32}m^2(5m+1)(5m-1)(84375m^4-12375m^2+436)x^8y^4\\
&-\frac{15}{8}m^2(5m-1)(3m+1)(5m+1)(5m+2)(75m^2+15m-4)x^8y^3\\
&+\frac{5}{48}m^2(3m+1)(5m+1)(5m+2)(15m+8)(225m^2+90m-13)x^8y^2\\
&-\frac{5}{56}m^2(3m+1)(5m+1)(5m+2)(5m+3)(15m+8)(15m+11)x^8y\\
&+\frac{1}{1344}m(3m+1)(5m+1)(5m+2)(5m+3)(15m+8)(15m+11)(15m+14)x^8\\
&+\frac{5}{56}m(15m-8)(15m-11)(5m-1)(5m-2)(5m-3)(3m-1)x^7y^7\\
&-\frac{25}{8}m^2(5m-1)(3m-1)(5m-2)(15m-8)^2x^7y^6\\
&+\frac{375}{8}m^2(5m-1)(5m-2)(3m-1)(45m^2-12m-2)x^7y^5\\
&-\frac{15}{8}m^2(5m+1)(5m-1)(5625m^3-2250m^2-75m+74)x^7y^4\\
&+\frac{15}{8}m^2(5m-1)(5m+1)(5625m^3+2250m^2-75m-74)x^7y^3\\
&-\frac{375}{8}m^2(3m+1)(5m+1)(5m+2)(45m^2+12m-2)x^7y^2\\
&+\frac{25}{8}m^2(3m+1)(5m+1)(5m+2)(15m+8)^2x^7y\\
&-\frac{5}{56}m(3m+1)(5m+1)(5m+2)(5m+3)(15m+8)(15m+11)x^7\\
&+\frac{5}{48}m(15m-8)(5m-1)(5m-2)(3m-1)(195m-107)x^6y^6\\
&-\frac{375}{8}m^2(39m-16)(5m-1)(5m-2)(3m-1)x^6y^5\\
&+\frac{5}{16}m^2(5m-1)(219375m^3-103500m^2+3675m+2342)x^6y^4\\
&-\frac{75}{4}m^2(5m-1)(5m+1)(975m^2-32)x^6y^3\\
&+\frac{5}{16}m^2(5m+1)(219375m^3+103500m^2+3675m-2342)x^6y^2\\
&-\frac{375}{8}m^2(3m+1)(5m+1)(5m+2)(39m+16)x^6y\\
&+\frac{5}{48}m(3m+1)(5m+1)(5m+2)(15m+8)(195m+107)x^6+15m(30m-13)(5m-1)(5m-2)(3m-1)x^5y^5\\
&-15m^2(5m-1)(2250m^2-1395m+218)x^5y^4+75m^2(5m-1)(900m^2-66m-23)x^5y^3\\
&-75m^2(5m+1)(900m^2+66m-23)x^5y^2+15m^2(5m+1)(2250m^2+1395m+218)x^5y\\
&-15m(3m+1)(5m+1)(5m+2)(30m+13)x^5+\frac{1}{2}m(5m-1)(10350m^2-6675m+1084)x^4y^4\\
&-15m^2(1380m-307)(5m-1)x^4y^3+5m^2(31050m^2-577)x^4y^2-15m^2(5m+1)(1380m+307)x^4y\\
&+\frac{1}{2}m(5m+1)(10350m^2+6675m+1084)x^4+45m(45m-11)(5m-1)x^3y^3-1125m^2(27m-4)x^3y^2\\
&+1125m^2(27m+4)x^3y-45m(5m+1)(45m+11)x^3+\frac{35}{2}m(105m-17)x^2y^2-3675m^2x^2y\\
&+\frac{35}{2}m(105m+17)x^2+120mxy-120mx+1. \qquad (7.24)
\end{aligned}
$$

8. The $F = M$ Conjecture. In this section we describe briefly the $F = M$ Conjecture of Armstrong [2] predicting a surprising relation between the M-triangle of the m-divisible partitions poset and the F-triangle of the generalised cluster complex of Fomin and Reading [13], we review the progress from [17], and we explain that our results from Sections 6 and 7 provide proofs of the $F = M$ Conjecture for E_7 and E_8. If we put this together with the results from [17], then it remains only the D_n case of the conjecture which is not completely proven.

For a non-negative integer m, the generalised cluster complex $\Delta^m(\Phi)$ is a certain simplicial complex on a certain set of "coloured" roots, the roots being from Φ. The precise definition will not be important here, we refer the reader to [13, Sec. 2]. The only fact which is important here is that some of the coloured roots can be positive, others negative. Let $f_{k,l}(\Phi, m)$ denote the number of faces of $\Delta^m(\Phi)$ which contain exactly k positive and l negative coloured roots. Define the F-triangle of $\Delta^m(\Phi)$, denoted by $F^m_\Phi(x, y)$, as the two-variable polynomial

$$F^m_\Phi(x, y) = \sum_{k,l\ge 0} f_{k,l}(\Phi, m)\, x^k y^l. \tag{8.1}$$

It is called "triangle" because all faces have cardinality at most n and, thus, in the summation in (8.1) we can restrict the summation indices to the triangle $k + l \le n$, $k, l \ge 0$.

The generalised version of Chapoton's (ex-)conjecture [11, Conjecture 1], due to Armstrong [2, Conjecture 5.3.2], is the following.

Conjecture FM. *For any finite root system Φ of rank n, we have*

$$F^m_\Phi(x, y) = y^n\, M^m_\Phi\left(\frac{1+y}{y-x}, \frac{y-x}{y}\right). \tag{8.2}$$

Equivalently,

$$(1 - xy)^n F^m_\Phi\left(\frac{x(1+y)}{1-xy}, \frac{xy}{1-xy}\right) = (M^m_\Phi)^*(-x, -y). \tag{8.3}$$

It is easy to see (cf. [17, Sec. 8]) that it is enough to prove the conjecture for the irreducible root systems. In [17], this has been done for the root systems of type A_n, B_n, $I_2(a)$, H_3, H_4, F_4, and E_6. The paper contains also a partial proof for the root system of type D_n. Given that we computed the M-triangle of the m-divisible non-crossing partitions poset for E_7 and E_8 and that the F-triangle of the generalised cluster complex has been computed in [17] for *all* irreducible root systems (thus, in particular, for E_7 and E_8), the verification of (8.3) for E_7 and E_8 is pure routine on a computer algebra system.

If we combine these observations with the proof of the $m = 1$ case of the $F = M$ Conjecture by Athanasiadis [3] (an alternative case-by-case proof is provided by the results in [17] and in the present paper), then we obtain the following theorem.

Theorem FM. *Conjecture FM is true with the possible exception of root systems containing a copy of the root system D_k for some k. If $m = 1$, Conjecture FM is true unconditionally.*

9. A reciprocity phenomenon. By staring at the explicit expressions for the M-triangles of the m-divisible non-crossing partitions posets for the irreducible root systems which we computed in [17] and in the present paper (in the D_n case, this expression is only conjectural), one observes a curious reciprocity relation between the original M-triangle and the M-triangle with m replaced by $-m$. From the outset, the M-triangle with $-m$ in place of m has no combinatorial meaning since there is no meaning for "$(-m)$-divisible non-crossing partitions." Nevertheless, it would be interesting to find an intrinsic explanation of the phenomenon given in the theorem below. In any case, examples of situations where combinatorial meaning was given to non-combinatorial parameters are not so uncommon; for example, there exist reciprocity theorems for *P-partitions* and the *Ehrhart quasi-polynomial* of polytopes (cf. [23, Theorems 4.5.7, 4.6.26, Cor. 4.5.15]), which have their explanation in Stanley's reciprocity theorem for linear homogeneous diophantine equations [23, Theorem 4.6.14], and there exist reciprocity theorems for monomer-dimer coverings of rectangles (cf. [1, 19, 22]).

Theorem 8. *For any finite root system of rank n, except possibly for those which contain a copy of D_k for some k, we have*

$$y^n M_\Phi^{-m}(xy, 1/y) = M_\Phi^m(x, y). \tag{9.1}$$

If the (conjectural) expression for $M_{D_n}^m(x, y)$ in [17, Sec. 11, Prop. D] *is correct, then* (9.1) *is true without any exceptions. Phrased differently, if the $F = M$ Conjecture is true, then* (9.1) *is true unconditionally.*

Formula (9.1) contains a reciprocity between numbers of maximal simplices in the generalised cluster complex $\Delta^m(\Phi)$, observed by Fomin and Reading [13, Eq. (11.2) and Prop. 11.4], and revealed as an instance of Ehrhart reciprocity by Athanasiadis and Tzanaki [5, Sec. 7], as a special case. (Again, for root systems containing a copy of D_k for some k, this is a conditional statement.) To see this, we first translate (9.1) into a reciprocity relation for the F-triangle $F_\Phi^m(x, y)$ via (8.2):

$$F_\Phi^m(x, y) = (1 + x)^n \, F_\Phi^{-m}\left(-\frac{x}{1 + x}, \frac{y - x}{1 + x}\right). \tag{9.2}$$

If we compare coefficients of $x^n y^0$ on both sides of this equation, then we obtain the relation

$$f_{n,0}(\Phi, m) = \sum_{k=0}^{n} (-1)^k f_k(\Phi, -m), \tag{9.3}$$

where $f_k(\Phi, m) = \sum_{l=0}^{k} f_{l,k-l}(\Phi, m)$ is the *total* number of faces in $\Delta^m(\Phi)$ containing exactly k roots (positive or negative). In the notation of [5, 13], we have $f_{n,0}(\Phi, m) = N^+(\Phi, m)$, and by [13, Eq. (10.2)] the sum on the right-hand side of (9.3) is equal to $(-1)^n f_n(\Phi, -m - 1)$ ($(-1)^n N(\Phi, -m - 1)$ in the notation of [13]). Thus, we arrive at the reciprocity relation

$$N^+(\Phi, m) = (-1)^n N(\Phi, -m - 1),$$

which is exactly [13, Eq. (11.2)].

The combinatorial significance of (9.2) in general is less clear. Comparison of coefficients of $x^k y^l$ on both sides leads to the identity

$$f_{k,l}(\Phi,m)=\sum_{r,s\ge 0}(-1)^{r+s+l}\binom{n-r-s}{k+l-r-s}\binom{s}{l}f_{r,s}(\Phi,-m). \tag{9.4}$$

10. A formula for the A_n decomposition numbers. Since the decomposition numbers $N_\Phi(T_1,T_2,\dots,T_d)$ carry so much information on the enumerative structure of (ordinary and generalised) non-crossing partitions (see, for example, Propositions 1 and 3), it would be of intrinsic interest to compute these numbers also for the classical root systems. As a matter of fact, the type A_n decomposition numbers of full rank are known due to a result of Goulden and Jackson [15, Theorem 3.2] on the minimal factorization of a long cycle. (The condition on the sum $l(\alpha_1)+l(\alpha_2)+\dots+l(\alpha_m)$ is misstated throughout the latter paper. It should be replaced by $l(\alpha_1)+l(\alpha_2)+\dots+l(\alpha_m)=(m-1)n+1$.) In the following theorem, we state this result in our language.

Theorem 9. *Let $\Phi=A_n$, and let $T_1,T_2,\dots,T_d$ be types with $\operatorname{rk}T_1+\operatorname{rk}T_2+\dots+\operatorname{rk}T_d=n$, where*

$$T_i=A_1^{m_1^{(i)}}*A_2^{m_2^{(i)}}*\dots*A_n^{m_n^{(i)}},\quad i=1,2,\dots,d.$$

Then

$$N_{A_n}(T_1,T_2,\dots,T_d)=(n+1)^{d-1}\prod_{i=1}^{d}\frac{1}{n-\operatorname{rk}T_i+1}\binom{n-\operatorname{rk}T_i+1}{m_1^{(i)},m_2^{(i)},\dots,m_n^{(i)}}, \tag{9.5}$$

where the multinomial coefficient is defined by

$$\binom{M}{m_1,m_2,\dots,m_n}=\frac{M!}{m_1!\,m_2!\cdots m_n!\,(M-m_1-m_2-\dots-m_n)!}.$$

This result allows one to derive a compact formula for *all* type A_n decomposition numbers.

Theorem 10. *Let $\Phi=A_n$, and let the types $T_1,T_2,\dots,T_d$ be given, where*

$$T_i=A_1^{m_1^{(i)}}*A_2^{m_2^{(i)}}*\dots*A_n^{m_n^{(i)}},\quad i=1,2,\dots,d.$$

Then

$$N_{A_n}(T_1,T_2,\dots,T_d)=(n+1)^{d-1}\binom{n+1}{\operatorname{rk}T_1+\operatorname{rk}T_2+\dots+\operatorname{rk}T_d+1}\\
\times\prod_{i=1}^{d}\frac{1}{n-\operatorname{rk}T_i+1}\binom{n-\operatorname{rk}T_i+1}{m_1^{(i)},m_2^{(i)},\dots,m_n^{(i)}}. \tag{9.6}$$

Proof. If we write r for $n-\operatorname{rk}T_1-\operatorname{rk}T_2-\cdots-\operatorname{rk}T_d$, then the relation (4.2) becomes

$$N_{A_n}(T_1,T_2,\dots,T_d)=\sum_{T:\operatorname{rk}T=r}N_{A_n}(T_1,T_2,\dots,T_d,T).$$

Upon letting $T=A_1^{m_1}*A_2^{m_2}*\cdots*A_n^{m_n}$, substitution of (9.5) in the above equation yields

$$N_{A_n}(T_1,T_2,\dots,T_d)=\sum_{m_1+2m_2+\cdots+nm_n=r}(n+1)^d\frac{1}{n-r+1}\binom{n-r+1}{m_1,m_2,\dots,m_n}\\
\cdot\prod_{i=1}^{d}\frac{1}{n-\operatorname{rk}T_i+1}\binom{n-\operatorname{rk}T_i+1}{m_1^{(i)},m_2^{(i)},\dots,m_n^{(i)}}.$$

Now, by comparison of coefficients of z^r on both sides of

$$(1+z+z^2+z^3+\cdots)^M=(1-z)^{-M},$$

we infer

$$\sum_{m_1+2m_2+\cdots+rm_r=r}\binom{M}{m_1,m_2,\dots,m_r}=\binom{M+r-1}{r}.$$

If we use this identity with $M=n-r+1$, we obtain our claim after little simplification. □

The reader should note that formula (9.6) generalises Kreweras' formula [18, Theorem 4] for the number of non-crossing partitions of $n+1$ with given block sizes, to which it reduces for $d=1$.

Appendix: The decomposition numbers for types A_n and D_n for small n, and for E_6. In this appendix we list the full rank decomposition numbers, that is, the decomposition numbers $N_\Phi(T_1,T_2,\dots,T_d)$, where $\operatorname{rk}T_1+\operatorname{rk}T_2+\cdots+\operatorname{rk}T_d=\operatorname{rk}\Phi$, for $\Phi=A_1,A_2,A_3,A_4,A_5,A_6,A_7,D_4,D_5,D_6,D_7,E_6$. These numbers are required for setting up the system of equations (4.3) in Sections 6 and 7. We do not list decomposition numbers which are zero. It was explained in Section 4 how to compute these numbers by setting up a system of linear equations for them. Whenever, aside from the listing of the numbers, there is no further comment, then the computation of the decomposition numbers is either trivial (for A_1 and A_2), or the equations in Propositions 2 and 3, together with the assignments from Propositions 4 and 6 determine them already uniquely. If not, then we mention the additional assignments, respectively considerations, which are required for the computation.[6] Clearly, the decomposition numbers for A_n, $n=1,2,\dots$, can be computed directly from Theorem 9.

The decomposition numbers for A_1. $N_{A_1}(A_1)=1$.

The decomposition numbers for A_2. $N_{A_2}(A_2)=1$, $N_{A_2}(A_1,A_1)=3$.

[6] The *Mathematica* inputs for the computations are available at
`http://www.mat.univie.ac.at/~kratt/artikel/cluster2.html`.

The decomposition numbers for A_3. $N_{A_3}(A_3)=1$, $N_{A_3}(A_2,A_1)=4$, $N_{A_3}(A_1^2,A_1)=2$, $N_{A_3}(A_1,A_1,A_1)=16$.

The decomposition numbers for A_4. $N_{A_4}(A_4)=1$, $N_{A_4}(A_3,A_1)=5$, $N_{A_4}(A_1{*}A_2,A_1)=5$, $N_{A_4}(A_2,A_2)=5$, $N_{A_4}(A_2,A_1^2)=5$, $N_{A_4}(A_1^2,A_1^2)=5$, $N_{A_4}(A_2,A_1,A_1)=25$, $N_{A_4}(A_1^2,A_1,A_1)=25$, $N_{A_4}(A_1,A_1,A_1,A_1)=125$.

The decomposition numbers for A_5. $N_{A_5}(A_5)=1$, $N_{A_5}(A_4,A_1)=6$, $N_{A_5}(A_1{*}A_3,A_1)=6$, $N_{A_5}(A_2^2,A_1)=3$, $N_{A_5}(A_3,A_2)=6$, $N_{A_5}(A_1*A_2,A_2)=12$, $N_{A_5}(A_1^3,A_2)=2$, $N_{A_5}(A_3,A_1^2)=9$, $N_{A_5}(A_1*A_2,A_1^2)=18$, $N_{A_5}(A_1^3,A_1^2)=3$, $N_{A_5}(A_3,A_1,A_1)=36$, $N_{A_5}(A_1*A_2,A_1,A_1)=72$, $N_{A_5}(A_1^3,A_1,A_1)=12$, $N_{A_5}(A_2,A_2,A_1)=36$, $N_{A_5}(A_2,A_1^2,A_1)=54$, $N_{A_5}(A_1^2,A_1^2,A_1)=81$, $N_{A_5}(A_2,A_1,A_1,A_1)=216$, $N_{A_5}(A_1^2,A_1,A_1,A_1)=324$, $N_{A_5}(A_1,A_1,A_1,A_1,A_1)=1296$.

The decomposition numbers for A_6. $N_{A_6}(A_6)=1$, $N_{A_6}(A_5,A_1)=7$, $N_{A_6}(A_1{*}A_4,A_1)=7$, $N_{A_6}(A_2*A_3,A_1)=7$, $N_{A_6}(A_4,A_2)=7$, $N_{A_6}(A_1*A_3,A_2)=14$, $N_{A_6}(A_2^2,A_2)=7$, $N_{A_6}(A_1^2*A_2,A_2)=7$, $N_{A_6}(A_4,A_1^2)=14$, $N_{A_6}(A_1*A_3,A_1^2)=28$, $N_{A_6}(A_2^2,A_1^2)=14$, $N_{A_6}(A_1^2*A_2,A_1^2)=14$, $N_{A_6}(A_3,A_3)=7$, $N_{A_6}(A_3,A_1*A_2)=21$, $N_{A_6}(A_3,A_1^3)=7$, $N_{A_6}(A_1*A_2,A_1*A_2)=63$, $N_{A_6}(A_1*A_2,A_1^3)=21$, $N_{A_6}(A_1^3,A_1^3)=7$, $N_{A_6}(A_4,A_1,A_1)=49$, $N_{A_6}(A_1*A_3,A_1,A_1)=98$, $N_{A_6}(A_2^2,A_1,A_1)=49$, $N_{A_6}(A_1^2*A_2,A_1,A_1)=49$, $N_{A_6}(A_3,A_2,A_1)=49$, $N_{A_6}(A_3,A_1^2,A_1)=98$, $N_{A_6}(A_1*A_2,A_2,A_1)=147$, $N_{A_6}(A_1*A_2,A_1^2,A_1)=294$, $N_{A_6}(A_1^3,A_2,A_1)=49$, $N_{A_6}(A_1^3,A_1^2,A_1)=98$, $N_{A_6}(A_3,A_1,A_1,A_1)=343$, $N_{A_6}(A_1*A_2,A_1,A_1,A_1)=1029$, $N_{A_6}(A_1^3,A_1,A_1,A_1)=343$, $N_{A_6}(A_2,A_2,A_2)=49$, $N_{A_6}(A_2,A_2,A_1^2)=98$, $N_{A_6}(A_2,A_1^2,A_1^2)=196$, $N_{A_6}(A_1^2,A_1^2,A_1^2)=392$, $N_{A_6}(A_2,A_2,A_1,A_1)=343$, $N_{A_6}(A_2,A_1^2,A_1,A_1)=686$, $N_{A_6}(A_1^2,A_1^2,A_1,A_1)=1372$, $N_{A_6}(A_2,A_1,A_1,A_1,A_1)=2401$, $N_{A_6}(A_1^2,A_1,A_1,A_1,A_1)=4802$, $N_{A_6}(A_1,A_1,A_1,A_1,A_1,A_1)=16807$.

The decomposition numbers for A_7. $N_{A_7}(A_7)=1$, $N_{A_7}(A_6,A_1)=8$, $N_{A_7}(A_1{*}A_5,A_1)=8$, $N_{A_7}(A_2*A_4,A_1)=8$, $N_{A_7}(A_3^2,A_1)=4$, $N_{A_7}(A_5,A_2)=8$, $N_{A_7}(A_1*A_4,A_2)=16$, $N_{A_7}(A_2*A_3,A_2)=16$, $N_{A_7}(A_1^2*A_3,A_2)=8$, $N_{A_7}(A_1*A_2^2,A_2)=8$, $N_{A_7}(A_5,A_1^2)=20$, $N_{A_7}(A_1{*}A_4,A_1^2)=40$, $N_{A_7}(A_2{*}A_3,A_1^2)=40$, $N_{A_7}(A_1^2{*}A_3,A_1^2)=20$, $N_{A_7}(A_1{*}A_2^2,A_1^2)=20$, $N_{A_7}(A_5,A_1,A_1)=64$, $N_{A_7}(A_1*A_4,A_1,A_1)=128$, $N_{A_7}(A_2*A_3,A_1,A_1)=128$, $N_{A_7}(A_1^2*A_3,A_1,A_1)=64$, $N_{A_7}(A_1*A_2^2,A_1,A_1)=64$, $N_{A_7}(A_4,A_3)=8$, $N_{A_7}(A_1*A_3,A_3)=24$, $N_{A_7}(A_2^2,A_3)=12$, $N_{A_7}(A_1^2*A_2,A_3)=24$, $N_{A_7}(A_1^4,A_3)=2$, $N_{A_7}(A_4,A_1*A_2)=32$, $N_{A_7}(A_1*A_3,A_1*A_2)=96$, $N_{A_7}(A_2^2,A_1*A_2)=48$, $N_{A_7}(A_1^2*A_2,A_1*A_2)=96$, $N_{A_7}(A_1^4,A_1*A_2)=8$, $N_{A_7}(A_4,A_1^3)=16$, $N_{A_7}(A_1*A_3,A_1^3)=48$, $N_{A_7}(A_2^2,A_1^3)=24$, $N_{A_7}(A_1^2*A_2,A_1^3)=48$, $N_{A_7}(A_1^4,A_1^3)=4$, $N_{A_7}(A_4,A_2,A_1)=64$, $N_{A_7}(A_1*A_3,A_2,A_1)=192$, $N_{A_7}(A_2^2,A_2,A_1)=96$, $N_{A_7}(A_1^2*A_2,A_2,A_1)=192$, $N_{A_7}(A_1^4,A_2,A_1)=16$, $N_{A_7}(A_4,A_1^2,A_1)=160$, $N_{A_7}(A_1{*}A_3,A_1^2,A_1)=480$, $N_{A_7}(A_2^2,A_1^2,A_1)=240$, $N_{A_7}(A_1^2{*}A_2,A_1^2,A_1)=480$, $N_{A_7}(A_1^4,A_1^2,A_1)=40$, $N_{A_7}(A_4,A_1,A_1,A_1)=512$, $N_{A_7}(A_1*A_3,A_1,A_1,A_1)=1536$, $N_{A_7}(A_2^2,A_1,A_1,A_1)=768$, $N_{A_7}(A_1^2*A_2,A_1,A_1,A_1)=1536$, $N_{A_7}(A_1^4,A_1,A_1,A_1)=128$, $N_{A_7}(A_3,A_3,A_1)=64$, $N_{A_7}(A_3,A_1*A_2,A_1)=256$, $N_{A_7}(A_3,A_1^3,A_1)=128$, $N_{A_7}(A_1*A_2,A_1*A_2,A_1)=1024$, $N_{A_7}(A_1*A_2,A_1^3,A_1)=512$, $N_{A_7}(A_1^3,A_1^3,A_1)=256$, $N_{A_7}(A_3,A_2,A_2)=64$, $N_{A_7}(A_3,A_2,A_1^2)=160$, $N_{A_7}(A_3,A_1^2,A_1^2)=400$, $N_{A_7}(A_1*A_2,A_2,A_2)=256$, $N_{A_7}(A_1*A_2,A_2,A_1^2)=640$, $N_{A_7}(A_1*A_2,A_1^2,A_1^2)=1600$, $N_{A_7}(A_1^3,A_2,A_2)=128$, $N_{A_7}(A_1^3,A_2,A_1^2)=320$, $N_{A_7}(A_1^3,A_1^2,A_1^2)=800$, $N_{A_7}(A_3,A_2,A_1,A_1)=512$, $N_{A_7}(A_3,A_1^2,A_1,A_1)=1280$, $N_{A_7}(A_1*A_2,A_2,A_1,A_1)=2048$, $N_{A_7}(A_1*A_2,A_1^2,A_1,A_1)=5120$,

$N_{A_7}(A_1^3, A_2, A_1, A_1) = 1024$, $N_{A_7}(A_1^3, A_1^2, A_1, A_1) = 2560$, $N_{A_7}(A_3, A_1, A_1, A_1, A_1) = 4096$, $N_{A_7}(A_1 * A_2, A_1, A_1, A_1, A_1) = 16384$, $N_{A_7}(A_1^3, A_1, A_1, A_1, A_1) = 8192$, $N_{A_7}(A_2, A_2, A_2, A_1) = 512$, $N_{A_7}(A_2, A_2, A_1^2, A_1) = 1280$, $N_{A_7}(A_2, A_1^2, A_1^2, A_1) = 3200$, $N_{A_7}(A_1^2, A_1^2, A_1^2, A_1) = 8000$, $N_{A_7}(A_2, A_2, A_1, A_1, A_1) = 4096$, $N_{A_7}(A_2, A_1^2, A_1, A_1, A_1) = 10240$, $N_{A_7}(A_1^2, A_1^2, A_1, A_1, A_1) = 25600$, $N_{A_7}(A_2, A_1, A_1, A_1, A_1, A_1) = 32768$, $N_{A_7}(A_1^2, A_1, A_1, A_1, A_1, A_1) = 81920$, $N_{A_7}(A_1, A_1, A_1, A_1, A_1, A_1, A_1) = 262144$.

The decomposition numbers for D_4. $N_{D_4}(D_4) = 1$, $N_{D_4}(A_3, A_1) = 9$, $N_{D_4}(A_1^3, A_1) = 3$, $N_{D_4}(A_2, A_2) = 6$, $N_{D_4}(A_2, A_1^2) = 9$, $N_{D_4}(A_2, A_1, A_1) = 36$, $N_{D_4}(A_1^2, A_1, A_1) = 27$, $N_{D_4}(A_1, A_1, A_1, A_1) = 162$.

The decomposition numbers for D_5. $N_{D_5}(D_5) = 1$, $N_{D_5}(D_4, A_1) = 4$, $N_{D_5}(A_4, A_1) = 8$, $N_{D_5}(A_1 * A_3, A_1) = 4$, $N_{D_5}(A_1^2 * A_2, A_1) = 4$, $N_{D_5}(A_3, A_2) = 12$, $N_{D_5}(A_1 * A_2, A_2) = 16$, $N_{D_5}(A_1^3, A_2) = 8$, $N_{D_5}(A_3, A_1^2) = 22$, $N_{D_5}(A_1 * A_2, A_1^2) = 8$, $N_{D_5}(A_1^3, A_1^2) = 4$, $N_{D_5}(A_3, A_1, A_1) = 80$, $N_{D_5}(A_1 * A_2, A_1, A_1) = 64$, $N_{D_5}(A_1^3, A_1, A_1) = 32$, $N_{D_5}(A_2, A_2, A_1) = 64$, $N_{D_5}(A_2, A_1^2, A_1) = 96$, $N_{D_5}(A_1^2, A_1^2, A_1) = 80$, $N_{D_5}(A_2, A_1, A_1, A_1) = 384$, $N_{D_5}(A_1^2, A_1, A_1, A_1) = 448$, $N_{D_5}(A_1, A_1, A_1, A_1, A_1) = 2048$.

The decomposition numbers for D_6. Unfortunately, the solution space of the system of linear equations does not consist of a unique solution but is two-dimensional. *Mathematica 5.2* expresses all the decomposition numbers in terms of $X = N_{D_6}(A_1^3, A_1^3)$ and $Y = N_{D_6}(A_1^4, A_1^2)$. In the sequel we make use of the two relations

$$N_{D_6}(A_1^2 * A_2, A_1^2) = 25 - \frac{27}{40}X - 3Y,$$
$$N_{D_6}(A_3, A_3) = 20 + \frac{9}{100}X.$$

They imply $X \equiv 0 \pmod{200}$, $X \le 1000/27 < 200$, $Y \equiv 2 \pmod 3$, and $Y \le 25/3 < 9$. Hence, we have $X = N_{D_6}(A_1^3, A_1^3) = 0$, and the three possibilities $Y = 2, 5, 8$. To decide which of the three values is the true value, we argue as in Section 6 when we had to decide which of two possible values for $N_{E_7}(A_1^2 * A_2, A_1^3)$ was the correct one (see the paragraph after (6.8)). Since Lemma 5 in the case of D_6 says that the orbits of reflections have all size 5, here the conclusion is that 5 must divide $Y = N_{D_6}(A_1^4, A_1^2)$. Thus, $N_{D_6}(A_1^4, A_1^2) = 5$. If we substitute this in the relations for the decomposition numbers found by *Mathematica*, then we obtain $N_{D_6}(D_6) = 1$, $N_{D_6}(D_5, A_1) = 5$, $N_{D_6}(A_5, A_1) = 10$, $N_{D_6}(A_1 * D_4, A_1) = 5$, $N_{D_6}(A_2 * A_3, A_1) = 5$, $N_{D_6}(A_1^2 * A_3, A_1) = 5$, $N_{D_6}(D_4, A_2) = 5$, $N_{D_6}(A_4, A_2) = 10$, $N_{D_6}(A_1 * A_3, A_2) = 30$, $N_{D_6}(A_2^2, A_2) = 10$, $N_{D_6}(A_1^2 * A_2, A_2) = 10$, $N_{D_6}(A_1^4, A_2) = 5$, $N_{D_6}(D_4, A_1^2) = 5$, $N_{D_6}(A_4, A_1^2) = 35$, $N_{D_6}(A_1 * A_3, A_1^2) = 30$, $N_{D_6}(A_2^2, A_1^2) = 10$, $N_{D_6}(A_1^2 * A_2, A_1^2) = 10$, $N_{D_6}(A_1^4, A_1^2) = 5$, $N_{D_6}(A_3, A_3) = 20$, $N_{D_6}(A_3, A_1 * A_2) = 45$, $N_{D_6}(A_3, A_1^3) = 25$, $N_{D_6}(A_1 * A_2, A_1 * A_2) = 70$, $N_{D_6}(A_1 * A_2, A_1^3) = 25$, $N_{D_6}(D_4, A_1, A_1) = 25$, $N_{D_6}(A_4, A_1, A_1) = 100$, $N_{D_6}(A_1 * A_3, A_1, A_1) = 150$, $N_{D_6}(A_2^2, A_1, A_1) = 50$, $N_{D_6}(A_1^2 * A_2, A_1, A_1) = 50$, $N_{D_6}(A_1^4, A_1, A_1) = 25$, $N_{D_6}(A_3, A_2, A_1) = 125$, $N_{D_6}(A_3, A_1^2, A_1) = 250$, $N_{D_6}(A_1 * A_2, A_2, A_1) = 250$, $N_{D_6}(A_1 * A_2, A_1^2, A_1) = 375$, $N_{D_6}(A_1^3, A_2, A_1) = 125$, $N_{D_6}(A_1^3, A_1^2, A_1) = 125$, $N_{D_6}(A_3, A_1, A_1, A_1) = 875$, $N_{D_6}(A_1 * A_2, A_1, A_1, A_1) = 1500$, $N_{D_6}(A_1^3, A_1, A_1, A_1) = 625$, $N_{D_6}(A_2, A_2, A_2) = 100$, $N_{D_6}(A_2, A_2, A_1^2) = 225$, $N_{D_6}(A_2, A_1^2, A_1^2) = 350$, $N_{D_6}(A_1^2, A_1^2, A_1^2) = 475$, $N_{D_6}(A_2, A_2, A_1, A_1) = 750$, $N_{D_6}(A_2, A_1^2,$

$A_1, A_1) = 1375$, $N_{D_6}(A_1^2, A_1^2, A_1, A_1) = 2000$, $N_{D_6}(A_2, A_1, A_1, A_1, A_1) = 5000$, $N_{D_6}(A_1^2, A_1, A_1, A_1, A_1) = 8125$, $N_{D_6}(A_1, A_1, A_1, A_1, A_1, A_1) = 31250$.

The decomposition numbers for D_7. In addition to the described equations and assignments, we use also the last assertion in Proposition 2 for the type A_1^5, that is, $N_{D_7}(A_1^5, A_2) = N_{D_7}(A_1^5, A_1^2) = 0$. Unfortunately, the solution space of the system of linear equations does not consist of a unique solution but is two-dimensional. *Mathematica 5.2* expresses all the decomposition numbers in terms of $X = N_{D_7}(A_1^4, A_1^3)$ and $Y = N_{D_7}(A_1^2 * A_2, A_1^3)$. In the sequel we make use of the two relations

$$N_{D_7}(A_1^4, A_1 * A_2) = \frac{6}{5}(36 - X), \tag{A.1}$$

$$N_{D_7}(A_1^2 * A_2, A_3) = \frac{3}{10}(Y + 162), \tag{A.2}$$

$$N_{D_7}(A_1 * A_3, A_1^3) = 84 - \frac{1}{3}Y, \tag{A.3}$$

$$N_{D_7}(A_1 * A_2^2, A_2) = \frac{14}{15}(36 - 3X - Y), \tag{A.4}$$

$$N_{D_7}(A_1 * A_2^2, A_1^2) = \frac{7}{5}(3X + Y - 36). \tag{A.5}$$

Relation (A.1) implies $X \equiv 1 \pmod 5$, while Relations (A.2) and (A.3) imply $Y \equiv 18 \pmod{30}$. Since decomposition numbers must be non-negative, Relations (A.4) and (A.5) force $3X + Y$ to be equal to 36. Hence, we must have $X = N_{D_7}(A_1^4, A_1^3) = 6$ and $Y = N_{D_7}(A_1^2 * A_2, A_1^3) = 18$. If we substitute this in the relations for the decomposition numbers found by *Mathematica*, then we obtain $N_{D_7}(D_7) = 1$, $N_{D_7}(D_6, A_1) = 6$, $N_{D_7}(A_6, A_1) = 12$, $N_{D_7}(A_1 * D_5, A_1) = 6$, $N_{D_7}(A_2 * D_4, A_1) = 6$, $N_{D_7}(A_1^2 * A_4, A_1) = 6$, $N_{D_7}(A_3^2, A_1) = 6$, $N_{D_7}(D_5, A_2) = 6$, $N_{D_7}(A_5, A_2) = 12$, $N_{D_7}(A_1 * D_4, A_2) = 12$, $N_{D_7}(A_1 * A_4, A_2) = 24$, $N_{D_7}(A_2 * A_3, A_2) = 36$, $N_{D_7}(A_1^2 * A_3, A_2) = 18$, $N_{D_7}(A_1^3 * A_2, A_2) = 12$, $N_{D_7}(D_5, A_1^2) = 9$, $N_{D_7}(A_5, A_1^2) = 54$, $N_{D_7}(A_1 * D_4, A_1^2) = 18$, $N_{D_7}(A_1 * A_4, A_1^2) = 36$, $N_{D_7}(A_2 * A_3, A_1^2) = 54$, $N_{D_7}(A_1^2 * A_3, A_1^2) = 27$, $N_{D_7}(A_1^3 * A_2, A_1^2) = 18$, $N_{D_7}(D_5, A_1, A_1) = 36$, $N_{D_7}(A_5, A_1, A_1) = 144$, $N_{D_7}(A_1 * D_4, A_1, A_1) = 72$, $N_{D_7}(A_1 * A_4, A_1, A_1) = 144$, $N_{D_7}(A_2 * A_3, A_1, A_1) = 216$, $N_{D_7}(A_1^2 * A_3, A_1, A_1) = 108$, $N_{D_7}(A_1^3 * A_2, A_1, A_1) = 72$, $N_{D_7}(D_4, A_3) = 6$, $N_{D_7}(A_4, A_3) = 18$, $N_{D_7}(A_1 * A_3, A_3) = 72$, $N_{D_7}(A_2^2, A_3) = 27$, $N_{D_7}(A_1^2 * A_2, A_3) = 54$, $N_{D_7}(A_1^4, A_3) = 18$, $N_{D_7}(D_4, A_1 * A_2) = 12$, $N_{D_7}(A_4, A_1 * A_2) = 72$, $N_{D_7}(A_1 * A_3, A_1 * A_2) = 180$, $N_{D_7}(A_2^2, A_1 * A_2) = 72$, $N_{D_7}(A_1^2 * A_2, A_1 * A_2) = 108$, $N_{D_7}(A_1^4, A_1 * A_2) = 36$, $N_{D_7}(D_4, A_1^3) = 2$, $N_{D_7}(A_4, A_1^3) = 60$, $N_{D_7}(A_1 * A_3, A_1^3) = 78$, $N_{D_7}(A_2^2, A_1^3) = 36$, $N_{D_7}(A_1^2 * A_2, A_1^3) = 18$, $N_{D_7}(A_1^4, A_1^3) = 6$, $N_{D_7}(D_4, A_2, A_1) = 36$, $N_{D_7}(D_4, A_1^2, A_1) = 54$, $N_{D_7}(A_4, A_2, A_1) = 144$, $N_{D_7}(A_4, A_1^2, A_1) = 432$, $N_{D_7}(A_1 * A_3, A_2, A_1) = 468$, $N_{D_7}(A_1 * A_3, A_1^2, A_1) = 918$, $N_{D_7}(A_2^2, A_2, A_1) = 180$, $N_{D_7}(A_2^2, A_1^2, A_1) = 378$, $N_{D_7}(A_1^2 * A_2, A_2, A_1) = 324$, $N_{D_7}(A_1^2 * A_2, A_1^2, A_1) = 486$, $N_{D_7}(A_1^4, A_2, A_1) = 108$, $N_{D_7}(A_1^4, A_1^2, A_1) = 162$, $N_{D_7}(D_4, A_1, A_1, A_1) = 216$, $N_{D_7}(A_4, A_1, A_1, A_1) = 1296$, $N_{D_7}(A_1 * A_3, A_1, A_1, A_1) = 3240$, $N_{D_7}(A_2^2, A_1, A_1, A_1) = 1296$, $N_{D_7}(A_1^2 * A_2, A_1, A_1, A_1) = 1944$, $N_{D_7}(A_1^4, A_1, A_1, A_1) = 648$, $N_{D_7}(A_3, A_3, A_1) = 216$, $N_{D_7}(A_3, A_1 * A_2, A_1) = 648$, $N_{D_7}(A_3, A_1^3, A_1) = 396$, $N_{D_7}(A_1 * A_2, A_1 * A_2, A_1) = 1728$, $N_{D_7}(A_1 * A_2, A_1^3, A_1) = 864$, $N_{D_7}(A_1^3, A_1^3, A_1) = 240$, $N_{D_7}(A_3, A_2, A_2) = 180$, $N_{D_7}(A_1 * A_2, A_2, A_2) = 576$, $N_{D_7}(A_1^3, A_2, A_2) = 384$, $N_{D_7}(A_3, A_2, A_1^2) = 486$, $N_{D_7}(A_1 *$

$A_2, A_2, A_1^2) = 1296$, $N_{D_7}(A_1^3, A_2, A_1^2) = 648$, $N_{D_7}(A_3, A_1^2, A_1^2) = 1053$, $N_{D_7}(A_1 * A_2, A_1^2, A_1^2) = 2592$, $N_{D_7}(A_1^3, A_1^2, A_1^2) = 1080$, $N_{D_7}(A_3, A_2, A_1, A_1) = 1512$, $N_{D_7}(A_3, A_1^2, A_1, A_1) = 3564$, $N_{D_7}(A_1 * A_2, A_2, A_1, A_1) = 4320$, $N_{D_7}(A_1 * A_2, A_1^2, A_1, A_1) = 9072$, $N_{D_7}(A_1^3, A_2, A_1, A_1) = 2448$, $N_{D_7}(A_1^3, A_1^2, A_1, A_1) = 4104$, $N_{D_7}(A_3, A_1, A_1, A_1, A_1) = 11664$, $N_{D_7}(A_1 * A_2, A_1, A_1, A_1, A_1) = 31104$, $N_{D_7}(A_1^3, A_1, A_1, A_1, A_1) = 15552$, $N_{D_7}(A_2, A_2, A_2, A_1) = 1296$, $N_{D_7}(A_2, A_2, A_1^2, A_1) = 3240$, $N_{D_7}(A_2, A_1^2, A_1^2, A_1) = 6804$, $N_{D_7}(A_1^2, A_1^2, A_1^2, A_1) = 13122$, $N_{D_7}(A_2, A_2, A_1, A_1, A_1) = 10368$, $N_{D_7}(A_2, A_1^2, A_1, A_1, A_1) = 23328$, $N_{D_7}(A_1^2, A_1^2, A_1, A_1, A_1) = 46656$, $N_{D_7}(A_2, A_1, A_1, A_1, A_1, A_1) = 77760$, $N_{D_7}(A_1^2, A_1, A_1, A_1, A_1, A_1) = 163296$, $N_{D_7}(A_1, A_1, A_1, A_1, A_1, A_1, A_1) = 559872$.

The decomposition numbers for E_6. These numbers have already been computed in [17, Sec. 17] using Stembridge's `coxeter` package [24]. An alternative, independent way to do this is by following the method described in Section 4. Here, in addition to the equations and assignments described at the beginning of the Appendix, we use also the last assertion in Proposition 2 for the type A_1^4, that is, $N_{E_6}(A_1^4, A_2) = N_{E_6}(A_1^4, A_1^2) = 0$. Unfortunately, the solution space of the system of linear equations does not consist of a unique solution but is one-dimensional. *Mathematica 5.2* expresses all the decomposition numbers in terms of $X = N_{E_6}(A_1^3, A_1^3)$. In the sequel we make use of the two relations

$$N_{E_6}(A_3, A_3) = \frac{1}{100}(2592 + 9X), \tag{A.6}$$

$$N_{E_6}(A_1^3, A_1^3) = \frac{1}{5}(192 - 6X). \tag{A.7}$$

From (A.6) we infer that $X \equiv 12 \pmod{100}$, while (A.7) implies that $X \leq 32$. Hence, we have $X = N_{E_6}(A_1^3, A_1^3) = 12$. If we substitute this in the relations for the decomposition numbers found by *Mathematica*, then we obtain $N_{E_6}(E_6) = 1$, $N_{E_6}(D_5, A_1) = 12$, $N_{E_6}(A_5, A_1) = 6$, $N_{E_6}(A_1 * A_4, A_1) = 12$, $N_{E_6}(A_1 * A_2^2, A_1) = 6$, $N_{E_6}(A_1^2 * A_2, A_2) = 36$, $N_{E_6}(D_4, A_2) = 4$, $N_{E_6}(A_4, A_2) = 24$, $N_{E_6}(A_1 * A_3, A_2) = 24$, $N_{E_6}(A_2^2, A_2) = 8$, $N_{E_6}(D_4, A_1^2) = 18$, $N_{E_6}(A_4, A_1^2) = 36$, $N_{E_6}(A_1 * A_3, A_1^2) = 36$, $N_{E_6}(A_1^2 * A_2, A_1^2) = 18$, $N_{E_6}(A_3, A_3) = 27$, $N_{E_6}(A_3, A_1 * A_2) = 72$, $N_{E_6}(A_3, A_1^3) = 36$, $N_{E_6}(A_1 * A_2, A_1 * A_2) = 48$, $N_{E_6}(A_1 * A_2, A_1^3) = 24$, $N_{E_6}(A_1^3, A_1^3) = 12$, $N_{E_6}(D_4, A_1, A_1) = 48$, $N_{E_6}(A_4, A_1, A_1) = 144$, $N_{E_6}(A_1 * A_3, A_1, A_1) = 144$, $N_{E_6}(A_2^2, A_1, A_1) = 24$, $N_{E_6}(A_1^2 * A_2, A_1, A_1) = 144$, $N_{E_6}(A_3, A_2, A_1) = 180$, $N_{E_6}(A_3, A_1^2, A_1) = 378$, $N_{E_6}(A_1 * A_2, A_2, A_1) = 336$, $N_{E_6}(A_1 * A_2, A_1^2, A_1) = 360$, $N_{E_6}(A_1^3, A_2, A_1) = 168$, $N_{E_6}(A_1^3, A_1^2, A_1) = 180$, $N_{E_6}(A_2, A_2, A_2) = 160$, $N_{E_6}(A_2, A_2, A_1^2) = 288$, $N_{E_6}(A_2, A_1^2, A_1^2) = 504$, $N_{E_6}(A_1^2, A_1^2, A_1^2) = 432$, $N_{E_6}(A_2, A_2, A_1, A_1) = 1056$, $N_{E_6}(A_2, A_1^2, A_1, A_1) = 1872$, $N_{E_6}(A_1^2, A_1^2, A_1, A_1) = 2376$, $N_{E_6}(A_3, A_1, A_1, A_1) = 1296$, $N_{E_6}(A_1 * A_2, A_1, A_1, A_1) = 1728$, $N_{E_6}(A_1^3, A_1, A_1, A_1) = 864$, $N_{E_6}(A_2, A_1, A_1, A_1, A_1) = 6912$, $N_{E_6}(A_1^2, A_1, A_1, A_1, A_1) = 10368$, $N_{E_6}(A_1, A_1, A_1, A_1, A_1, A_1) = 41472$.

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Notes. After the first version of this paper was distributed, Eleni Tzanaki found a uniform proof of Armstrong's $F = M$ Conjecture (presented here in Conjecture FM) in "Faces of generalized cluster complexes and noncrossing partitions," `arχiv:math.CO/0605785`. Thus, our Theorem 8 becomes an unconditional theorem, that is to say, the reciprocity relation (9.1) holds for any finite root system Φ. Furthermore, the explicit form of the M-triangle in type D_n is given by [17, Prop. D] up to a simple substitution of variables.

The problem of computing the decomposition numbers for the type B_n is solved implicitly in "Enumeration of m-ary cacti" (Adv. Appl. Math. **24** (2000), 22–56) by Miklós Bóna, Michel Bousquet, Gilbert Labelle and Pierre Leroux. This is explained in detail in the article "Decomposition numbers for finite Coxeter groups and generalised non-crossing partitions" by Thomas Müller and the author, which also solves the problem for the remaining type D_n. In particular, the results in that paper, together with the results in the present paper and in [17], constitute an independent — case-by-case — proof of the $F = M$ Conjecture.

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